Univerzitet u Zenici
Filozofski fakultet
Odsjek: Matematika i informatika
Akademska 2013/2014.

Sadržaj sveske sa vježbi iz
Analize III

(II dio sveske - sadrži gradivo od 8 do 15 sedmice)

Literatura i zbirke za dodatno usavršavanje:

- Vajzović, Malenica: Diferencijalni račun funkcija više promjenljivih;
- Vajzović, Malenica: Integralni račun funkcija više promjenljivih;
- Perić, Tomić, Karačić: Zbirka riješenih zadataka iz matematike II;
- Demidovič: Zadaci i riješeni primjeri iz više matematike s primjenom na tehničke nauke;
- Ljaško i ostali: Zbirka zadataka iz matematičke analize II;
- Miličić, Uščumlić: Zbirka zadataka iz više matematike II;
- Berman: Sbornik zadach po kursu matematicheskogo analiza;
- Fatkić, Dragičević: Diferencijalni račun funkcija dviju i više promjenjivih;

Dio tablice izvoda

1) $(c)' = 0;$

2) $(u+v-w)' = u'+v'-w';$

3) $(uv)' = u'v+v'u;$

3a) $(cu)' = cu';$

4) $\left(\frac{u}{v}\right)' = \frac{u'v-v'u}{v^2};$

4a) $\left(\frac{u}{c}\right)' = \frac{u'}{c};$

4б) $\left(\frac{c}{v}\right)' = -\frac{cv'}{v^2};$

5) $(x^n)' = nx^{n-1};$

6) $(\sin x)' = \cos x;$

7) $(\cos x)' = -\sin x;$

8) $(\operatorname{tg} x)' = \sec^2 x;$

9) $(\operatorname{ctg} x)' = -\operatorname{cosec}^2 x.$

5) $(u^n)' = nu^{n-1}\cdot u';$

8) $(\operatorname{tg} u)' = \sec^2 u\cdot u';$

6) $(\sin u)' = \cos u\cdot u';$

9) $(\operatorname{ctg} u)' = -\operatorname{cosec}^2 u\cdot u'.$

7) $(\cos u)' = -\sin u\cdot u';$

10) $(a^u)' = a^u \ln a\cdot u';$

11) $(\log u)' = \frac{u'}{u}\log e;$

10a) $(e^u)' = e^u u';$

11a) $(\ln u)' = \frac{u'}{u};$

10б) $(a^x)' = a^x \ln a;$

11б) $(\log x)' = \frac{1}{x}\log e;$

10в) $(e^x)' = e^x;$

11в) $(\ln x)' = \frac{1}{x}.$

12) $(\arcsin u)' = \frac{u'}{\sqrt{1-u^2}};$

12a) $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}};$

13) $(\arccos u)' = -\frac{u'}{\sqrt{1-u^2}};$

13a) $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}};$

14) $(\operatorname{arctg} u)' = \frac{u'}{1+u^2};$

14a) $(\operatorname{arctg} x)' = \frac{1}{1+x^2};$

15) $(\operatorname{arcctg} u)' = -\frac{u'}{1+u^2};$

15a) $(\operatorname{arcctg} x)' = -\frac{1}{1+x^2}.$

Dio tablice integrala

1. $\int u^a\,du = \frac{u^{a+1}}{a+1}+C,\ a\neq -1.$

2. $\int u^{-1}\,du = \int\frac{du}{u} = \int\frac{u'}{u}\,dx = \ln|u|+C.$

3. $\int a^u\,du = \frac{a^u}{\ln a}+C;\ \int e^u\,du = e^u+C.$

4. $\int \sin u\,du = -\cos u+C.$

5. $\int \cos u\,du = \sin u+C.$

6. $\int \sec^2 u\,du = \operatorname{tg} u+C.$

7. $\int \operatorname{cosec}^2 u\,du = -\operatorname{ctg} u+C.$

8. $\int\frac{du}{u^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{u}{a}+C.$

9. $\int\frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C.$

10. $\int\frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a}+C.$

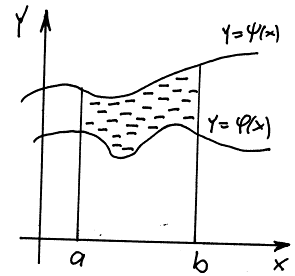
11. $\int\frac{du}{\sqrt{u^2+a}} = \ln\left|u+\sqrt{u^2+a}\right|+C.$

Trostruki integral

$I = \iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz$, Ω oblast integracije u prostoru

ako je $\Omega: \begin{cases} a \le x \le b \\ \varphi(x) \le y \le \psi(x) \\ \alpha(x,y) \le z \le \beta(x,y) \end{cases}$ tada

$$I = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} dy \int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz$$



Oblast Ω možemo projicirati na

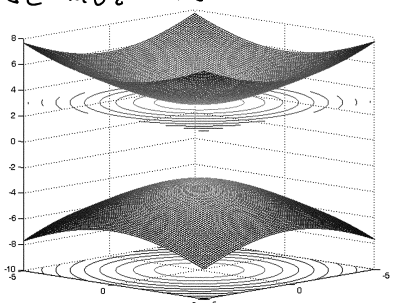
a) xoy ravan ili

b) yoz ravan ili

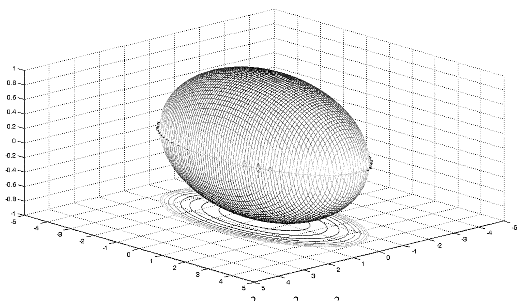
c) xoz ravan

U gornjem primjeru Ω smo prvo projicirali na xoy ravan.

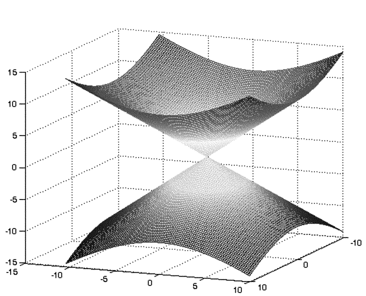
I se može izraziti na 6 načina.



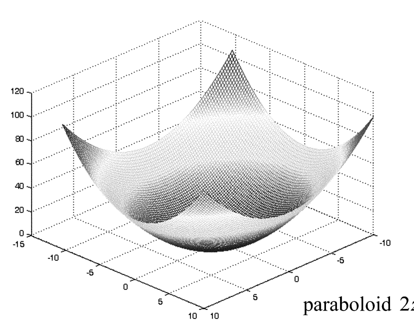
hiperboloid $x^2 + y^2 - z^2 = -9$



elipsoid $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1$



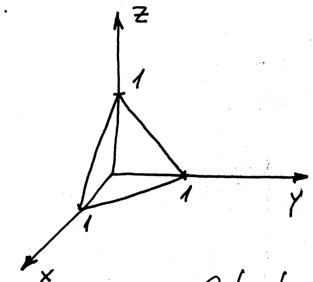
čunj $x^2 + y^2 = z^2$



paraboloid $2z = x^2 + y^2$

\# Izračunajte $\iiint\limits_{\Omega} (1-x)yz\,dx\,dy\,dz$ gdje je Ω oblast ograničena ravnima $x=0$, $y=0$, $z=0$ i $x+y+z=1$

Rj.



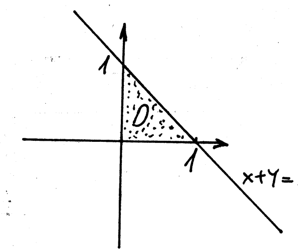
$x+y+z=1 \Leftrightarrow \frac{x}{1} + \frac{y}{1} + \frac{z}{1} = 1$ segmentni oblik jedn. ravni

$x=0$ je yoz ravan

$y=0$ je xoz ravan

$z=0$ je xoy ravan

Odredimo projekciju oblasti na xoy ravan



$$\begin{aligned} x+y+z&=1 \\ z&=0 \\ \hline x+y&=1 \end{aligned}$$

$z = 1-x-y$

Sa slike odredimo granice

$0 \le x \le 1$

$0 \le y \le 1-x$

$0 \le z \le 1-x-y$

$$\iiint\limits_{\Omega} (1-x)yz\,dx\,dy\,dz = \int_0^1 (1-x)\,dx \int_0^{1-x} y\,dy \int_0^{1-x-y} z\,dz = \int_0^1 (1-x)\,dx \int_0^{1-x} y \cdot \frac{1}{2} z^2 \Big|_0^{1-x-y} dy$$

$$= \frac{1}{2} \int_0^1 (1-x)\,dx \int_0^{1-x} y\,(\underbrace{1-x}_{a} - \underbrace{y}_{b})^2\,dy = \frac{1}{2} \int_0^1 (1-x)\,dx \int_0^{1-x} y\left[(1-x)^2 - 2y(1-x) + y^2\right] dy$$

$$= \frac{1}{2} \int_0^1 (1-x)\,dx \int_0^{1-x} \left[(1-x)^2 y - 2y^2(1-x) + y^3\right] dy = \frac{1}{2} \int_0^1 (1-x)\left[(1-x)^2 \frac{1}{2} y^2 \Big|_0^{1-x} -\right.$$

$$\left. - 2 \cdot \frac{1}{3} y^3 \Big|_0^{1-x} \cdot (1-x) + \frac{1}{4} y^4 \Big|_0^{1-x} \right] dx = \frac{1}{2} \int_0^1 (1-x)\left[(1-x)^4 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)\right] dx$$

$$= \frac{1}{2} \cdot \frac{1}{12} \int_0^1 (1-x)^5\,dx = \left| \begin{matrix} 1-x = t \\ -dx = dt \\ dx = -dt \end{matrix} \quad \begin{matrix} x=0 \Rightarrow t=1 \\ x=1 \Rightarrow t=0 \end{matrix} \right| = -\frac{1}{24} \int_1^0 t^5\,dt = -\frac{1}{24} \cdot \frac{1}{6} t^6 \Big|_1^0 = \frac{1}{144}$$

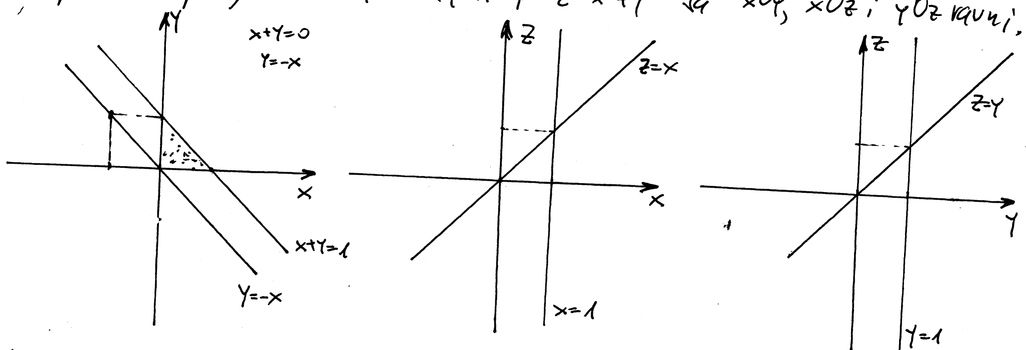
Izračunati trojni integral

$$I=\iiint\limits_G \frac{1}{(1+z)^3}\,dx\,dy\,dz$$

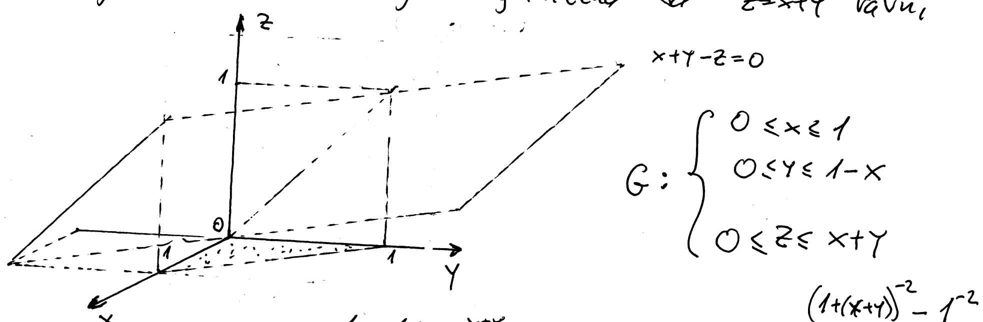
gdje je oblast G u I oktantu ograničena ravnima

$x+y=1,\quad z=x+y,\quad x=0,\quad y=0,\quad z=0.$

Rj. Napravimo presjek ravni $x+y=1$ i $z=x+y$ sa xOy, xOz i yOz ravni.



Iz presjeka vidimo da je ravan $x+y=1$ paralelna sa z osom a da je oblast G odozgo ograničena sa $z=x+y$ ravni



$$G:\begin{cases} 0\le x\le 1\\ 0\le y\le 1-x\\ 0\le z\le x+y\end{cases}$$

$$I=\iiint\limits_G \frac{1}{(1+z)^2}\,dx\,dy\,dz=\int_0^1 dx\int_0^{1-x} dy\int_0^{x+y}(1+z)^{-3}\,d(1+z)=\frac{1}{-2}\int_0^1 dx\int_0^{1-x}\overbrace{(1+z)^{-2}\Big|_0^{x+y}}^{(1+(x+y))^{-2}-1^{-2}}dy=$$

$$=-\frac{1}{2}\int_0^1 dx\int_0^{1-x}(1+x+y)^{-2}\,d(1+x+y)-\left(-\frac{1}{2}\right)\int_0^1 dx\int_0^{1-x} dy=-\frac{1}{2}\int_0^1(-1)(1+x+y)^{-1}\Big|_0^{1-x}dx+$$

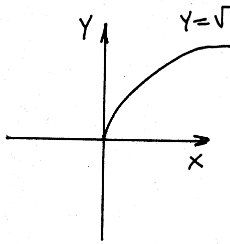
$$+\frac{1}{2}\int_0^1(1-x)\,dx=\frac{1}{2}\int_0^1\underbrace{\left(2^{-1}-(1+x)^{-1}\right)}_{\frac{1}{2}-(1+x)^{-1}}dx+\frac{1}{2}\int_0^1(1-x)\,dx=$$

$$=\frac{1}{2}\left(\frac{1}{2}x\Big|_0^1-\ln(1+x)\Big|_0^1+x\Big|_0^1-\frac{1}{2}x^2\Big|_0^1\right)=$$

$$=\frac{1}{2}\left(\frac{1}{2}-\ln 2+1-\frac{1}{2}\right)=\frac{1}{2}(1-\ln 2)$$ traženo rješenje

Izračunati $I=\iiint\limits_{\Omega} y\cos(x+z)\,dx\,dy\,dz$ gdje je Ω oblast ograničena plohom $y=\sqrt{x}$ i ravnima $y=0$, $z=0$ i $x+z=\frac{\pi}{2}$.

Rj.

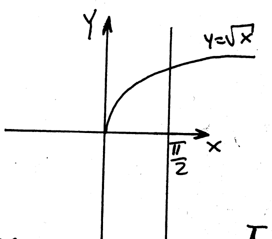


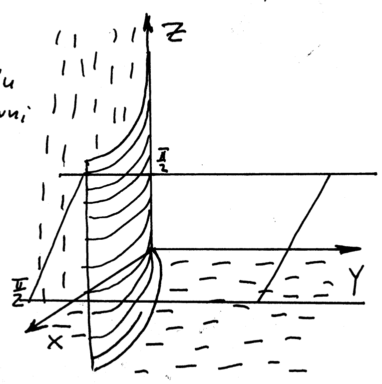
$y=0$ je xOz ravan

$z=0$ je xOy ravan

$x+z=\frac{\pi}{2}$

za $z=0$ dobiću projekciju ove ravni na xoy ravan





$0\le x\le\frac{\pi}{2}$

$0\le y\le\sqrt{x}$

$0\le z\le\frac{\pi}{2}-x$

$$I=\int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}}y\,dy\int_0^{\frac{\pi}{2}-x}\cos(x+z)\,dz=\int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}}y\sin(x+z)\Big|_{z=0}^{z=\frac{\pi}{2}-x}dy\overset{(*)}{=}$$

$$\int\cos(x+a)\,dx=\left|\begin{matrix}x+a=t\\dx=dt\end{matrix}\right|=\int\cos t\,dt=\sin t+c=\sin(x+a)+c$$

$$\overset{(*)}{=}\int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}}y\Big[\underbrace{\sin\left(x+\frac{\pi}{2}-x\right)}_{1}-\sin x\Big]dy=\int_0^{\frac{\pi}{2}}\underbrace{\frac{1}{2}y^2\Big|_0^{\sqrt{x}}}_{x-0}(1-\sin x)\,dx=$$

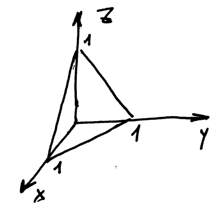
$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}x\,dx-\frac{1}{2}\int_0^{\frac{\pi}{2}}x\sin x\,dx=\left|\begin{matrix}u=x & dv=\sin x\,dx\\du=dx & v=-\cos x\end{matrix}\right|=\frac{1}{2}\cdot\frac{1}{2}x^2\Big|_0^{\frac{\pi}{2}}-$$

$$-\frac{1}{2}\left[-x\cos x\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos x\,dx\right]=\frac{1}{4}\cdot\frac{\pi^2}{4}-\frac{1}{2}\sin x\Big|_0^{\frac{\pi}{2}}=\frac{\pi^2-8}{16}$$

Izračunati trostruki integral $I=\iiint\limits_{\Omega}\frac{dx\,dy\,dz}{(x+y+z+1)^3}$, ako je Ω oblast ograničena koordinatnim ravnima i ravni $x+y+z=1$.

Rj. $x+y+z=1$ je ravan koja na koordinatnim osama prolazi kroz tačke $(1,0,0)$, $(0,1,0)$ i $(0,0,1)$

$$\Omega:\begin{cases}0\le x\le 1\\0\le y\le 1-x\\0\le z\le 1-x-y\end{cases}$$



$$I=\int_0^1dx\int_0^{1-x}dy\int_0^{1-x-y}\frac{dz}{(x+y+z+1)^3}\overset{(*)}{=}$$

$$\int\frac{dz}{(x+y+z+1)^3}=\left|\begin{matrix}x+y+z+1=t\\dz=dt\end{matrix}\right|=\int\frac{dt}{t^3}=\int t^{-3}dt=\frac{t^{-2}}{-2}+c=$$

$$=\frac{-1}{2(x+y+z+1)^2}+c$$

$$\overset{(*)}{=}\int_0^1dx\int_0^{1-x}\frac{-1}{2(x+y+z+1)^2}\Big|_0^{1-x-y}dy=\int_0^1dx\int_0^{1-x}\Big(\frac{-1}{2(x+y+1-x-y+1)^2}-$$

$$-\frac{-1}{2(x+y+0+1)^2}\Big)dy=\int_0^1dx\int_0^{1-x}\left(-\frac{1}{8}+\frac{1}{2(x+y+1)^2}\right)dy=$$

$$=-\frac{1}{2}\int_0^1dx\int_0^{1-x}\left(\frac{1}{4}-\frac{1}{(x+y+1)^2}\right)dy\overset{(**)}{=}-\frac{1}{2}\int_0^1\left(\frac{1}{4}y\Big|_0^{1-x}+\frac{1}{x+y+1}\Big|_0^{1-x}\right)dx$$

$$\int\frac{dy}{(x+y+1)^2}=\left|\begin{matrix}x+y+1=t\\dy=dt\end{matrix}\right|=\int\frac{dt}{t^2}=\frac{t^{-1}}{-1}+c=\frac{-1}{t}+c=\frac{-1}{x+y+1}+c\quad\ldots(**)$$

$$=-\frac{1}{2}\int_0^1\left(\frac{1}{4}(1-x)+\frac{1}{2}-\frac{1}{x+1}\right)dx=-\frac{1}{2}\Big(\overbrace{\frac{1}{4}x\Big|_0^1-\frac{1}{4}\cdot\frac{x^2}{2}\Big|_0^1}^{\frac{1}{8}}+\frac{1}{2}x\Big|_0^1-\underbrace{\ln|x+1|\Big|_0^1}_{\ln 2-\ln 1}\Big)=\frac{1}{2}\ln 2-\frac{5}{16}$$

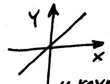
(#) Izračunati trostruki integral $I=\iiint\limits_{\Omega} z\,dx\,dy\,dz$, ako je

$\Omega:\ y=x,\ \ y=2x,\ \ 2x=1,\ \ x^2+y^2+z^2=1,\ \ z\geq 0$

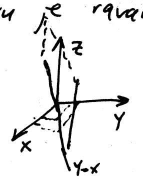
(oblast Ω je ograničena ovim površinama).

Rj: Komentarišimo površi koje čine Ω.

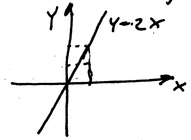
$y=x$ u ravni je prava



$y=x$ u prostoru je ravan koja sadrži pravu $y=x$

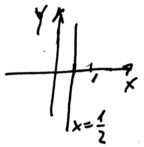


$y=2x$ u ravni je prava

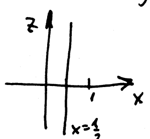


$y=2x$ u prostoru je ravan koja u ravni xOy sadrži pravu $y=2x$

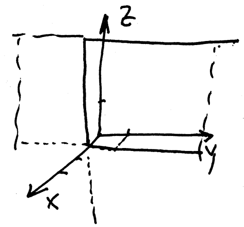
$2x=1$ u ravni xOy je prava



u ravni xOz je isto prava



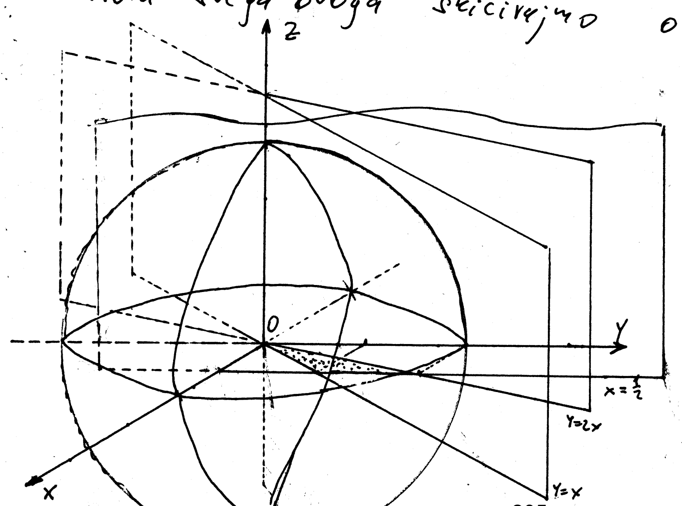
U prostoru to je ravan koja sadrži u xOz ravni pravu $x=\frac{1}{2}$ i u xOy ravni pravu $x=\frac{1}{2}$



$x=\frac{1}{2}$ je ravan koja je paralelna sa yOz osom

$x^2+y^2+z^2=1$ je jednačina kružnice

Na osnovu svega ovoga skicirajmo oblast Ω.



Oblast Ω je kružni isječak čija projekcija na xOy ravan je predstavljena tačkama na slici. Možemo zaključiti

$$\Omega:\begin{cases} 0\leq x\leq \frac{1}{2}\\ x\leq y\leq 2x\\ 0\leq z\leq \sqrt{1-x^2-y^2}\end{cases}$$

$$I=\iiint\limits_{\Omega} z\,dx\,dy\,dz=\int_0^{\frac{1}{2}}dx\int_x^{2x}dy\int_0^{\sqrt{1-x^2-y^2}}z\,dz=\int_0^{\frac{1}{2}}dx\int_x^{2x}\frac{1}{2}z^2\Big|_0^{\sqrt{1-x^2-y^2}}dy=$$

$$=\frac{1}{2}\int_0^{\frac{1}{2}}dx\int_x^{2x}(1-x^2-y^2)\,dy=\frac{1}{2}\int_0^{\frac{1}{2}}\left(y\Big|_x^{2x}-x^2y\Big|_x^{2x}-\frac{1}{3}y^3\Big|_x^{2x}\right)dx=$$

$$=\frac{1}{2}\int_0^{1/2}\left(x-x^3-\frac{1}{3}7x^3\right)dx=\frac{1}{2}\int_0^{1/2}\left(x-\frac{10}{3}x^3\right)dx=\frac{1}{2}\left(\frac{1}{2}x^2\Big|_0^{\frac{1}{2}}-\frac{\overset{5}{\cancel{10}}}{3}\cdot\frac{1}{\underset{2}{\cancel{4}}}x^4\Big|_0^{\frac{1}{2}}\right)$$

$$=\frac{1}{2}\left(\frac{1}{2}\cdot\frac{1}{4}-\frac{5}{6}\cdot\frac{1}{16}\right)=\frac{1}{2}\left(\frac{1}{8}-\frac{5}{\;}\right)=\frac{1}{2}\cdot\frac{12-5}{96}=\frac{7}{192}$$

1. Izračunaj trostruki integral $I=\int_{-1}^{1}dx\int_{x^2}^{1}dy\int_{0}^{2}(4+z)dz$.

Rješenje:

$$I=\int_{-1}^{1}dx\int_{x^2}^{1}dy\int_{0}^{2}(4+z)dz=\int_{-1}^{1}dx\int_{x^2}^{1}4z\Big|_{0}^{2}+\frac{z^2}{2}\Big|_{0}^{2}dy=\int_{-1}^{1}dx\int_{x^2}^{1}(8+2)dy=10\int_{-1}^{1}y\Big|_{x^2}^{1}dx=$$

$$=10\int_{-1}^{1}(1-x^2)dx=10(x\Big|_{-1}^{1}-\frac{x^3}{3}\Big|_{-1}^{1})=10(2-\frac{2}{3})=10\cdot\frac{4}{3}=\frac{40}{3}$$

2. Izračunaj trostruki integral $\iiint\limits_{G}\frac{dxdydz}{1-x-y}$, gdje je G ograničena ravnima :

a) x+y+z=1, x=0, y=0, z=0;

b) x=0, x=1, y=2, y=5, z=2, z=4.

Rješenja:

a) $\iiint\limits_{G}\frac{dxdydz}{1-x-y}$ x=0, y=0, z=0

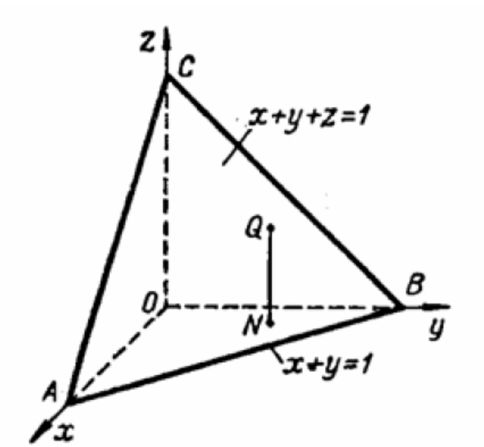
Skicirajmo oblast G (vidi sliku desno).

x+y+z=1 $\Leftrightarrow \frac{x}{1}+\frac{y}{1}+\frac{z}{1}=1$

x= 0 je yOz ravan

y= 0 je xOz ravan

z= 0 je xOy ravan



Odredimo projekciju oblasti na xOy ravan:
Nacrtati sliku (uputa: pogledati xoy ravan sa slike desno).

x+y+z=1
z= 0

x+y=1
z= 1-x-y

Sa slike projekcije odredimo granice: $0\le x\le 1$, $0\le y\le 1-x$, $0\le z\le 1-x-y$

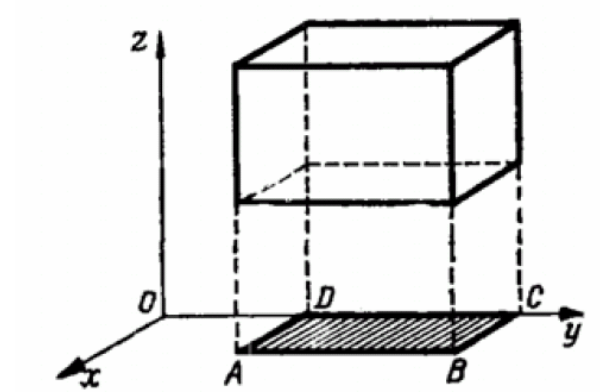
$$\iiint\limits_{G}\frac{dxdydz}{1-x-y}=\int_{0}^{1}dx\int_{0}^{1-x}\frac{dy}{1-x-y}\int_{0}^{1-x-y}dz=\int_{0}^{1}dx\int_{0}^{1-x}\left(\frac{1}{1-x-y}\cdot z\Big|_{0}^{1-x-y}\right)dy=$$

$$=\int_{0}^{1}dx\int_{0}^{1-x}(\frac{1}{1-x-y}\cdot(1-x-y))dy=\int_{0}^{1}dx\int_{0}^{1-x}dy=\int_{0}^{1}y\Big|_{0}^{1-x}dx=\int_{0}^{1}(1-x)dx=$$

$$=x\Big|_{0}^{1}-\frac{x^2}{2}\Big|_{0}^{1}=1-\frac{1}{2}=\frac{1}{2}$$

b) $\iiint\limits_{G}\frac{dxdydz}{1-x-y}$ x=0, x=1, y=2, y=5, z=2, z=4.

Skicirajmo oblast G (vidi sliku).



$$\int_{0}^{1}dx\int_{2}^{5}\frac{dy}{1-x-y}\int_{2}^{4}dz=\int_{0}^{1}dx\int_{2}^{5}z\Big|_{2}^{4}\frac{dy}{1-x-y}=2\int_{0}^{1}dx\int_{2}^{5}\frac{dy}{1-x-y}=\left|\begin{array}{l}1-x-y=t\\ -dy=dt\\ y=2\Rightarrow t=-1-x\\ y=5\Rightarrow t=-4-x\end{array}\right|=$$

$$-2\int_{0}^{1}dx\int_{-1-x}^{-4-x}\frac{dt}{t}=-2\int_{0}^{1}\ln|t|\Big|_{-1-x}^{-4-x}=-2\int_{0}^{1}\{\ \ln[-(4-x)]-\ln(-1-x)\ \}dx=$$

$$=-2\int_{0}^{1}\ln|x+4|dx+2\int_{0}^{1}\ln|x+1|dx\overset{*}{=}$$

Zadaci za vježbu

U zadacima **3474. — 3476.** **proceniti date integrale.**

3474. $\iiint\limits_{\Omega} (x^2+y^2+z^2)\,dv$, gde je Ω—lopta $x^2+y^2+z^2<R^2$.

3475. $\iiint\limits_{\Omega} (x+y+z)\,dv$, gde je Ω—lopta $x\geqslant 1,\ y>1,\ z>1,\ x<3,$ $y<3,\ z<3.$

3476. $\iiint\limits_{\Omega} (x+y-z+10)\,dv$, gde je Ω—lopta $x^2+y^2+z^2<3.$

U zadacima 3517 — 3524 izračunati navedene trostruke i trojne integrale

3517. $\int\limits_0^1 dx \int\limits_0^2 dy \int\limits_0^3 dz.$ **3518.** $\int\limits_0^a dx \int\limits_0^b dy \int\limits_0^c (x+y+z)\,dz.$

3519. $\int\limits_0^a dx \int\limits_0^x dx \int\limits_0^y xyz\,dz.$ **3520.** $\int\limits_0^a dx \int\limits_0^x dy \int\limits_0^{xy} x^3y^2z\,dz.$

3521. $\int\limits_0^{e-1} dx \int\limits_0^{e-x-1} dy \int\limits_e^{x+y+e} \frac{\ln(z-x-y)}{(x-e)(x+y-e)}\,dz.$

3522. $\iiint\limits_{\Omega} \frac{dx\,dy\,dz}{(x+y+z+1)^3}$, Ω je oblast ograničena ravnima $x=0,\ y=0,$ $z=0,\ x+y+z=1.$

3523. $\iiint\limits_{\Omega} xy\,dx\,dy\,dz$, Ω je oblast ograničena hiperboličnim paraboloidom $z=xy$ i ravnima $x+y=1$ i $z=0$ $(z\geq 0)$.

3524. $\iiint\limits_{\Omega} y\cos(z+x)\,dx\,dy\,ds$, Ω je oblast ograničena cilindrom $y=\sqrt{x}$ i ravnima $y=0,\ z=0$ i $x+z=\frac{\pi}{2}$.

Rješenja

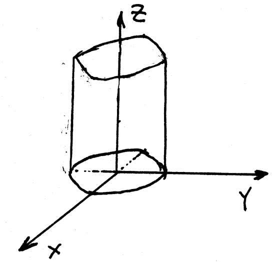
3474. $0<I<\frac{4}{3}\pi R^5.$ **3475.** $24<I<72.$

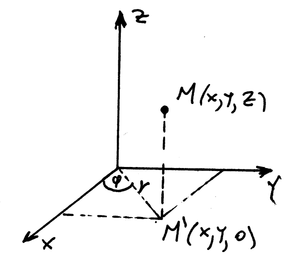
3476. $29\pi\sqrt{3}<I<52\pi\sqrt{3}.$ **3517.** 6. **3518.** $\frac{abc(a+b+c)}{2}.$ **3519.** $\frac{a^6}{48}.$ **3520.** $\frac{a^{11}}{110}.$

3521. $2e-5.$ **3522.** $\frac{1}{2}\left(\ln 2-\frac{5}{8}\right).$ **3523.** $\frac{1}{180}.$ **3524.** $\frac{\pi^2}{16}-\frac{1}{2}.$

Računanje trostrukih integrala uvođenjem cilindričnih i sfernih koordinata

cilindrične koordinate





uvodimo smjenu

$x = r\cos\varphi$

$y = r\sin\varphi$

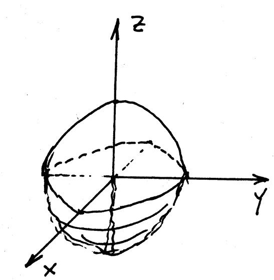
$z = z$

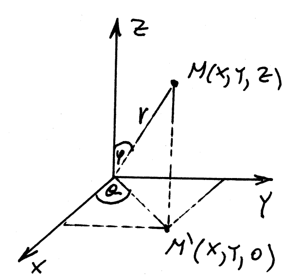
$dx\,dy\,dz = r\,dr\,d\varphi\,dz$

cilindrične koordinate obično uvedemo ako se pojavi izraz x^2+y^2 $\left(x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2(\sin^2\varphi+\cos^2\varphi) = r^2\right)$

$(r\geq 0,\quad 0\leq\varphi\leq 2\pi)$

sferne koordinate





uvodimo smjenu

$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

$dx\,dy\,dz = r^2\sin\varphi\,dr\,d\varphi\,d\theta$

$r\geq 0$

$0\leq\varphi\leq\pi$

$0\leq\theta\leq 2\pi$

$x^2+y^2+z^2 = r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\varphi\sin^2\theta + r^2\cos^2\varphi = \ldots = r^2$

sferne koordinate obično uvodimo ako se u podintegralnoj f-ji ili u opisu oblasti integracije pojavljuje izraz $x^2+y^2+z^2$.

Dati trojni integral $\iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz$

transformisati na trostruki u cilindričnim koordinatama (sa određenim posebnim granicama integracije) ako je Ω oblast u prvom oktantu ograničen cilindrom $x^2+y^2=R^2$ i ravnima $z=0$, $z=1$, $y=x$ i $y=x\sqrt{3}$

Rj.

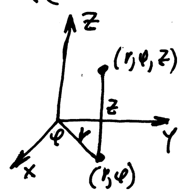
Cilindrične koordinate glase

$x=r\cos\varphi$

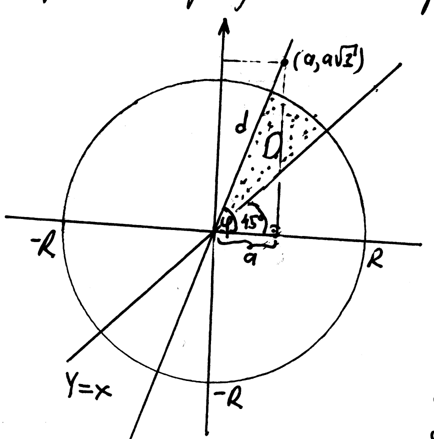
$y=r\sin\varphi$

$z=z$

$dx\,dy\,dz=r\,dr\,d\varphi\,dz$



Napravimo presjek datih površina sa xOy ravni:



$\cos\varphi=\dfrac{a}{d}=\dfrac{a}{2a}=\dfrac{1}{2}\Rightarrow\varphi=\dfrac{\pi}{3}$

$d^2=a^2+3a^2=4a^2$

$d=2a$

$\Omega \xrightarrow{\text{transformise}} \Omega'$

$$\Omega':\begin{cases}0\le r\le R\\ \frac{\pi}{4}\le\varphi\le\frac{\pi}{3}\\ 0\le z\le 1\end{cases}$$

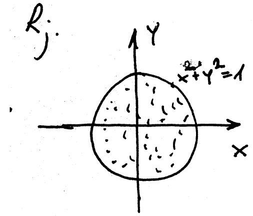
Sad nije teško vidjeti da je

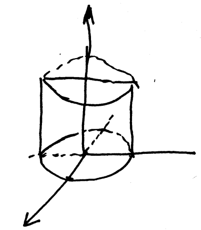
$$\iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz=\int_0^1 dz\int_{\pi/4}^{\pi/3} d\varphi\int_0^R f(r\cos\varphi, r\sin\varphi, z)\,r\,dr.$$

Izračunati $I=\iiint\limits_{\Omega}(x^2+y^2+z)^3\,dx\,dy\,dz$ gdje je

Ω oblast ograničena sa $x^2+y^2=1$, $z=0$ i $z=1$.

Rj.





uvodimo smjene:

$x=r\cos\varphi$

$y=r\sin\varphi$

$z=z$

$x^2+y^2=r^2\cos^2\varphi+r^2\sin^2\varphi=r^2$

$x^2+y^2=1$, $r^2=1$, $r\ge 0$, $0\le r\le 1$

$0\le z\le 1$, $0\le\varphi\le 2\pi$

$$\Omega'=\begin{cases}0\le r\le 1\\ 0\le\varphi\le 2\pi\\ 0\le z\le 1\end{cases}\qquad dx\,dy\,dz=r\,dr\,d\varphi\,dz$$

$$I=\iiint\limits_{\Omega}(x^2+y^2+z)^3\,dx\,dy\,dz=\iiint\limits_{\Omega'}(r^2+z)^3 r\,dr\,d\varphi\,dz=$$

$$=\int_0^1 dz\int_0^{2\pi}d\varphi\int_0^1(r^2+z)^3\cdot r\,dr=\left|\begin{matrix}r^2+z=t & r=0\Rightarrow t=z\\ 2r\,dr=dt & r=1\Rightarrow t=z+1\\ r\,dr=\frac{1}{2}dt & \end{matrix}\right|$$

$$=\frac{1}{2}\int_0^1 dz\int_0^{2\pi}d\varphi\int_z^{z+1}t^3\,dt=\frac{1}{2}\int_0^1 dz\int_0^{2\pi}\frac{1}{4}t^4\Big|_z^{z+1}d\varphi=\frac{1}{8}\int_0^1\left[(z+1)^4-z^4\right]\varphi\Big|_0^{2\pi}dz$$

$$=\frac{1}{8}\cdot 2\pi\int_0^1\left[(z+1)^4-z^4\right]dz=\frac{\pi}{4}\cdot\left(\frac{1}{5}(z+1)^5\Big|_0^1-\frac{1}{5}z^5\Big|_0^1\right)=$$

$$\int(z+1)^4dz=\left|\begin{matrix}t=z+1\\ dt=dz\end{matrix}\right|=\int t^4\,dt=\frac{1}{5}t^5+c=\frac{1}{5}(z+1)^5+c$$

$$=\frac{\pi}{20}(31-1)=\frac{30\pi}{20}=\frac{3\pi}{2}$$

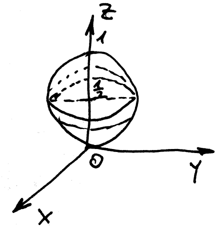
Izračunati $I=\iiint\limits_{\Omega}\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$ gdje je Ω oblast ograničena sferom $x^2+y^2+z^2=z$.

Rj.

$$x^2+y^2+z^2=z$$
$$x^2+y^2+z^2-z=0$$
$$x^2+y^2+z^2-2\cdot z\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2=0$$
$$x^2+y^2+\left(z-\frac{1}{2}\right)^2=\frac{1}{4}$$

centar sfere u tački $S(0,0,\frac{1}{4})$

poluprečnik sfere $r=\frac{1}{2}$



uvodimo smjene

$$x=r\sin\varphi\cos\theta$$
$$y=r\sin\varphi\sin\theta$$
$$z=r\cos\varphi$$

Odredimo granice za r, φ i θ nove oblasti

$$x^2+y^2+z^2=r^2 \quad \text{iz } x^2+y^2+z^2=z$$
$$r^2=r\cos\varphi \quad /:r \quad (r\neq 0)$$
$$r=\cos\varphi \quad \text{kako je } r>0 \Rightarrow \cos\varphi>0 \quad \text{tj. } 0\le\varphi\le\frac{\pi}{2}$$

$$\Omega':\begin{cases}0\le r\le\cos\varphi\\ 0\le\varphi\le\frac{\pi}{2}\\ 0\le\theta\le 2\pi\end{cases} \qquad dx\,dy\,dz=r^2\sin\varphi\,dr\,d\varphi\,d\theta$$

$$I=\iiint\limits_{\Omega}\sqrt{x^2+y^2+z^2}\,dx\,dy\,dz=\iiint\limits_{\Omega'}\sqrt{r^2}\,r^2\sin\varphi\,dr\,d\varphi\,d\theta=$$

$$\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}\sin\varphi\,d\varphi\int_0^{\cos\varphi}r^3\,dr=\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\varphi\,\frac{1}{4}r^4\Big|_0^{\cos\varphi}d\varphi=$$

$$=\frac{1}{4}\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\varphi\cos^4\varphi\,d\varphi=\left|\begin{matrix}\cos\varphi=t & \varphi=0\Rightarrow t=1\\ -\sin\varphi\,d\varphi=dt & \varphi=\frac{\pi}{2}\Rightarrow t=0\\ \sin\varphi\,d\varphi=-dt & \end{matrix}\right|=$$

$$=\frac{1}{4}\int_0^{2\pi}d\theta\int_0^{1}t^4\,dt=\frac{1}{4}\int_0^{2\pi}\frac{1}{5}t^5\Big|_0^1 d\theta=\frac{1}{20}\,\theta\Big|_0^{2\pi}=\frac{1}{20}\cdot 2\pi=\frac{\pi}{10}$$

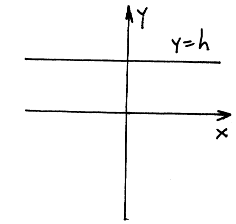
Izračunati trostruki integral

$$K=\iiint\limits_{T}y\,dx\,dy\,dz$$

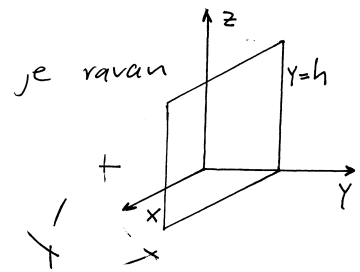
gdje je oblast T ograničena površinama $y=\sqrt{x^2+z^2}$ i $y=h$, $h>0$.

Rj. Pokušajmo skicirati oblast T.

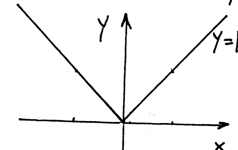
U xOy-ravni $y=h$ je prava.



U prostoru $y=h$ je ravan

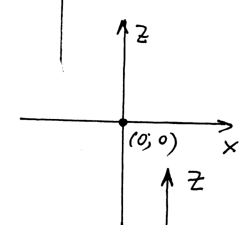


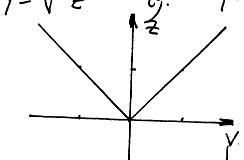
U xOy-ravni površina $y=\sqrt{x^2+z^2}$ je oblika $y=\sqrt{x^2}$



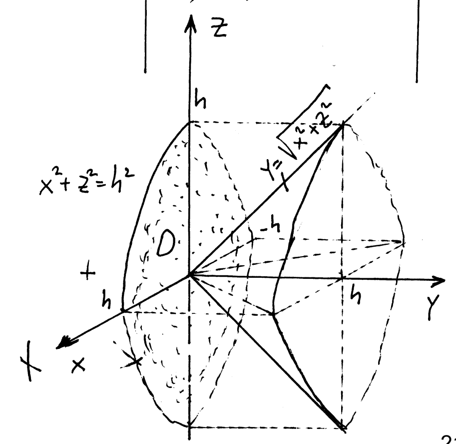
U xOz-ravni površina $y=\sqrt{x^2+z^2}$ je oblika $0=\sqrt{x^2+z^2}$ tj. tačka $(0;0)$.

U yOz-ravni površina $y=\sqrt{x^2+z^2}$ je oblika $y=\sqrt{z^2}$ tj. $y=|z|$.





Ako napravimo presjek površina $y=\sqrt{x^2+z^2}$ i $y=h$ dobićemo $h=\sqrt{x^2+z^2}$ tj. $x^2+z^2=h^2$ (krug poluprečnika h)



Oblast T (pola čunja) je prikazan na slici lijevo. Ako napravimo projekciju oblasti T na xOz ravan dobićemo sljedeće granice:

$$T:\begin{cases} -h\le x\le h\\ -\sqrt{h^2-z^2}\le z\le\sqrt{h^2-z^2}\\ \sqrt{x^2+z^2}\le y\le h\end{cases}$$

Pomoću pravougaonih koordinata dati trostruki integral je teško izračunati.

Uvodimo cilindrične koordinate i to

$x = r\cos\varphi$

$z = r\sin\varphi$

$y = y$

$dx\,dy\,dz = r\,dr\,d\varphi\,dy$

$$T \xrightarrow{\text{transformiše}} T':\begin{cases} 0\le r\le h\\ 0\le\varphi\le 2\pi\\ r\le y\le h\end{cases}$$

Prema tome

$$K=\iiint_T y\,dx\,dy\,dz = \left|\begin{matrix}\text{uvodimo}\\ \text{cilindrične}\\ \text{koordinate}\end{matrix}\right| = \iiint_{T'} y\,r\,dr\,d\varphi\,dy =$$

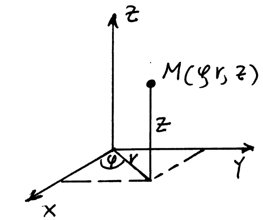
$$=\int_0^{2\pi}d\varphi\int_0^h r\,dr\int_r^h y\,dy = \int_0^{2\pi}d\varphi\int_0^h r\,\underbrace{\frac{1}{2}y^2\Big|_r^h}_{h^2-r^2}dr =$$

$$=\frac{1}{2}\int_0^{2\pi}d\varphi\int_0^h(rh^2-r^3)\,dr = \frac{1}{2}\int_0^{2\pi}\Big(\underbrace{\frac{1}{2}r^2h^2\Big|_0^h}_{\frac{1}{2}r^4} - \underbrace{\frac{1}{4}r^4\Big|_0^h}_{-\frac{1}{4}h^4}\Big)d\varphi$$

$$=\frac{1}{2}\cdot\frac{1}{4}h^4\int_0^{2\pi}d\varphi = \frac{1}{8}h^4\varphi\Big|_0^{2\pi} = \frac{h^4\pi}{4}$$ traženo rješenje

(#) Dat je trostruki integral $\int_0^{2\pi}d\varphi\int_0^2 r^3 dr\int_0^{\sqrt{4-r^2}}dz$ u cilindričnim koordinatama. Skicirati oblast integracije i izračunati taj integral prelazeći na sferne koordinate.

Rj. U cilindričnim koordinatama proizvoljna tačka M je opisana na sljedeći način



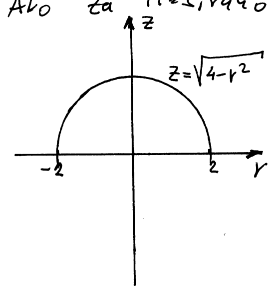
$$\Omega:\begin{cases} 0\le\varphi\le 2\pi\\ 0\le r\le 2\\ 0\le z\sqrt{4-r^2}\end{cases}$$

Na osnovu izgleda oblasti Ω vidimo da je projekcija figure na xOy ravan oblika

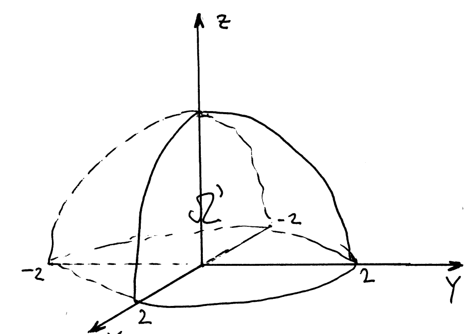
$$D:\begin{cases} 0\le\varphi\le 2\pi\\ 0\le r\le 2\end{cases}$$

tj. krug sa centrom u koordinatnom početku poluprečnika 2.

Ako za fiksirano φ posmatramo rOz ravan imamo



Prema tome oblast integracije Ω je polulopta



Cilindrične koordinate glase

$x = r\cos\varphi$

$y = r\sin\varphi$

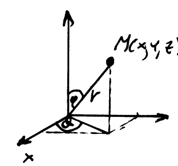
$z = z$

$dx\,dy = r\,dr\,d\varphi$

Tako da bi prelaskom na pravougaone koordinate sad imali

$$\int_0^{2\pi} d\varphi \int_0^2 r^3 dr \int_0^{\sqrt{1-r^2}} dz = \iiint_{\Omega} r^2\, r\,dr\,d\varphi\, dz = \left| \begin{array}{l} \text{prelazimo na pravougaone koordinate} \\ \Omega \xrightarrow{\text{transformiše}} \Omega' \\ r\,dr\,d\varphi = dx\,dy \\ r^2 = r^2(\sin^2\varphi+\cos^2\varphi) \\ \quad = r^2\sin^2\varphi + r^2\cos^2\varphi \\ \quad = (r\sin\varphi)^2+(r\cos\varphi)^2 \\ \quad = x^2+y^2 \end{array}\right| =$$

$$= \iiint_{\Omega'} (x^2+y^2)\, dx\,dy\,dz .$$



Sferne koordinate glase

$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

$dx\,dy\,dz = r^2\sin\varphi\, dr\,d\varphi\,d\theta$

$$\Omega' \xrightarrow{\text{transformiše}} \Omega'' : \begin{cases} 0 \le r \le 2 \\ 0 \le \varphi \le \pi/2 \\ 0 \le \theta \le 2\pi \end{cases} \qquad x^2+y^2 = r^2\sin^2\varphi$$

$$\iiint_{\Omega'} (x^2+y^2)\,dx\,dy\,dz = \left|\begin{array}{l}\text{uvodimo} \\ \text{sferne} \\ \text{koordinate}\end{array}\right| = \iiint_{\Omega''} r^2\sin^2\varphi\; r^2\sin\varphi\, dr\,d\theta\,d\varphi =$$

$$= \int_0^{2\pi} d\theta \int_0^2 r^4 dr \int_0^{\pi/2} \sin^3\varphi\, d\varphi \overset{(x)}{=} \theta\Big|_0^{2\pi} \cdot \frac{1}{5}r^5\Big|_0^2 \cdot \frac{2}{3} = \frac{2^7}{15}\pi$$

traženo rješenje

$$\int_0^{\pi/2} \sin^3\varphi\, d\varphi = \int_0^{\pi/2} \sin\varphi\underbrace{(1-\cos^2\varphi)}_{\sin^2\varphi}\, d\varphi = \left|\begin{array}{l} d(\sin\varphi) = \cos\varphi\, d\varphi \\ d(\cos\varphi) = -\sin\varphi\, d\varphi \end{array}\right| = -\int_0^{\pi/2} (1-\cos^2\varphi)\, d(\cos\varphi)$$

$$= -\left(\cos\varphi\Big|_0^{\pi/2} - \frac{1}{3}\cos^3\varphi\Big|_0^{\pi/2}\right) = -\left((0-1) - \frac{1}{3}(0-1)\right) = -\left(-1+\frac{1}{3}\right) = \frac{2}{3} \quad \dots(x)$$

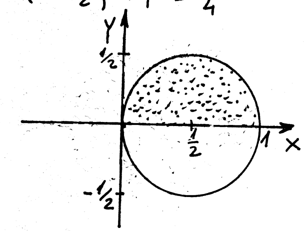
Izračunati integral $\iiint_\Omega \sqrt{z\cdot(x^2+y^2)}\, dx\,dy\,dz$ gdje je Ω oblast $x^2+y^2 \le x$; $y \ge 0$, $z \ge 0$, $z \le 3$.

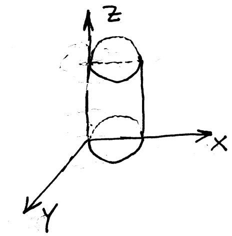
Rj. U ravni xOy kako izgleda $x^2+y^2 \le x$?

$x^2 - x + y^2 = 0$

$x^2 - 2\cdot\frac{1}{2}\cdot x + \frac{1}{4} + y^2 = \frac{1}{4}$

$(x-\frac{1}{2})^2 + y^2 = \frac{1}{4}$





Uvodimo smjenu

$x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

$x^2+y^2 \le x$

$r^2\cos^2\varphi + r^2\sin^2\varphi \le r\cos\varphi$

$r^2 \le r\cos\varphi \quad /:r \quad (r \ne 0)$

$r \le \cos\varphi$

kako je $r \ge 0$ to je $\cos\varphi \ge 0$

$y \ge 0$

$r\sin\varphi \ge 0 \quad /:r$

$\sin\varphi \ge 0$

imam $0 \le r \le \cos\varphi$

$\sin\varphi \ge 0$

$\cos\varphi \ge 0$

$0 \le z \le 3$

$$\Rightarrow \quad \Omega' : \begin{cases} 0 \le r \le \cos\varphi \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le z \le 3 \end{cases} \qquad dx\,dy\,dz = J\,d\varphi\,dr\,dz$$

$$\iiint_\Omega \sqrt{z(x^2+y^2)}\,dx\,dy\,dz = \iiint_{\Omega'} \sqrt{z r^2}\; r\, d\varphi\,dr\,dz = \int_0^3 dz \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\cos\varphi} \sqrt{z}\, r^2\, dr =$$

$$= \int_0^3 \sqrt{z}\,dz \int_0^{\frac{\pi}{2}} \frac{1}{3} r^3\Big|_0^{\cos\varphi} d\varphi = \frac{1}{3}\int_0^3 \sqrt{z}\,dz \int_0^{\frac{\pi}{2}} \cos^3\varphi\, d\varphi = \frac{1}{3}\int_0^3 \sqrt{z}\,dz \int_0^{\frac{\pi}{2}} \cos\varphi(1-\sin^2\varphi)\,d\varphi$$

$$= \left|\begin{array}{ll} \sin\varphi = t & \varphi = 0 \Rightarrow t = 0 \\ \cos\varphi\, d\varphi = dt & \varphi = \frac{\pi}{2} \Rightarrow t = 1 \end{array}\right| = \frac{1}{3}\int_0^3 \sqrt{z}\,dz \int_0^1 (1-t^2)\,dt = \frac{1}{3}\int_0^3 \sqrt{z}\left(t\Big|_0^1 - \frac{1}{3}t^3\Big|_0^1\right)dz$$

$$= \frac{1}{3}\left(1-\frac{1}{3}\right)\frac{z^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^3 = \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot 3^{\frac{3}{2}} = \frac{4}{27}\sqrt{3^3} = \frac{4}{9}\sqrt{3}$$

Izračunati trostruki integral $J=\iiint\limits_W (x^2+y^2+z^2)\,dx\,dy\,dz$

gdje je oblast W ograničena površinom $3(x^2+y^2)+z^2=3a^2$.

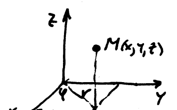
Rj. Skicirajmo oblast W

$3(x^2+y^2)+z^2=3a^2$

$3x^2+3y^2+z^2=3a^2 \quad /:3a^2$

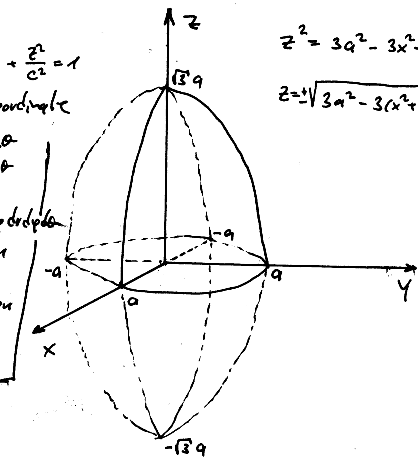
$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{3a^2}=1$

jednačina elipse



za elipsu $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ uopštene sferne koordinate glase $x=ar\sin\varphi\cos\theta$, $y=br\sin\varphi\sin\theta$, $z=cr\cos\varphi$, $dx\,dy\,dz=abc\,r^2\sin\varphi\,dr\,d\varphi\,d\theta$

U ovom slučaju uopštene sferne koordinate ne mogu na lagan način riješiti zadatak



$z^2=3a^2-3x^2-3y^2$

$z=\pm\sqrt{3a^2-3(x^2+y^2)}$

Uvodimo cilindrične koordinate

$x=r\cos\varphi$

$y=r\sin\varphi$

$z=z$

$dx\,dy\,dz=r\,dr\,d\varphi\,dz$

$x^2+y^2+z^2=r^2+z^2$

$W \xrightarrow{\text{transformiše}} W': \begin{cases} 0\le r\le a \\ 0\le\varphi\le 2\pi \\ -\sqrt{3(a^2-(x^2+y^2))}\le z\le\sqrt{3(a^2-(x^2+y^2))} \\ -\sqrt{3(a^2-r^2)}\le z\le\sqrt{3(a^2-r^2)} \end{cases}$

$$J=\iiint\limits_W (x^2+y^2+z^2)\,dx\,dy\,dz=\left|\text{uvodimo cilindrične koordinate}\right|=\iiint\limits_{W'}(r^2+z^2)\,r\,dr\,d\varphi\,dz=$$

$$=\int_0^{2\pi}d\varphi\int_0^a dr\int_{-\sqrt{3(a^2-r^2)}}^{\sqrt{3(a^2-r^2)}}(r^2+z^2)\,r\,dz=\int_0^{2\pi}d\varphi\int_0^a\left(r^2 z\Big|_{-\sqrt{3(a^2-r^2)}}^{\sqrt{3(a^2-r^2)}}+\frac{z^3}{3}\Big|_{-\sqrt{3(a^2-r^2)}}^{\sqrt{3(a^2-r^2)}}\right)r\,dr$$

$$=\int_0^{2\pi}d\varphi\int_0^a\left(r^2\,2\sqrt3\sqrt{a^2-r^2}+\frac13\left(3\sqrt3\sqrt{(a^2-r^2)^3}+3\sqrt3\sqrt{(a^2-r^2)^3}\right)\right)r\,dr=$$

ako ovo izvučemo ispred zagrade

$$=\int_0^{2\pi}d\varphi\int_0^a\left(2\sqrt3\,r^2\sqrt{a^2-r^2}+2\sqrt3\,(a^2-r^2)\sqrt{a^2-r^2}\right)r\,dr=2\sqrt3\,a^2\int_0^{2\pi}d\varphi\int_0^a\sqrt{a^2-r^2}\,r\,dr$$

$$=\left|d(a^2-r^2)=-2r\,dr\right|=-\sqrt3\,a^2\int_0^{2\pi}d\varphi\int_0^a\sqrt{a^2-r^2}\,d(a^2-r^2)=\ \dots\ =\frac{1}{\sqrt3}4\pi a^5$$

rješenje

Izračunati $I=\iiint\limits_\Omega\sqrt{x^2+y^2}\,dx\,dy\,dz$ gdje je Ω oblast

$x^2+y^2+z^2\le1$ i $x^2+y^2+z^2\le 2z$.

Rj. $x^2+y^2+z^2\le1$

je unutrašnjost sfere poluprečnika 1 sa centrom u tački $(0,0,0)$

$x^2+y^2+z^2\le 2z$

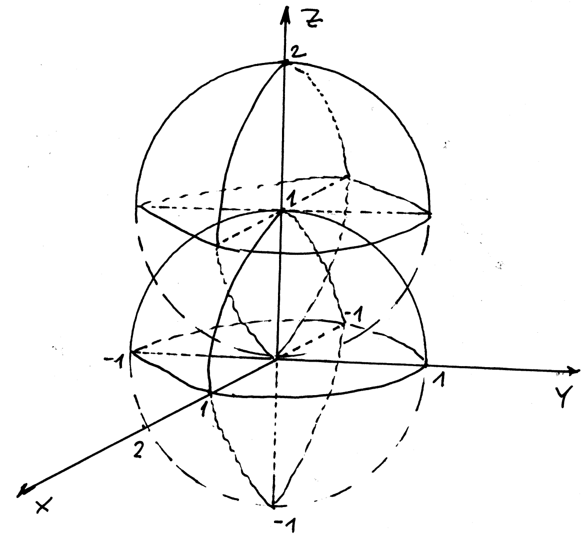
$x^2+y^2+z^2-2z\le0$

$x^2+y^2+z^2-2\cdot z\cdot1+1\le1$

$x^2+y^2+(z-1)^2\le1$

unutrašnjost sfere poluprečnika 1 sa centrom u tački $(0,0,1)$

Skicirajmo dvije date sfere



opišimo oblast integracije uz pomoć sfernih koordinata

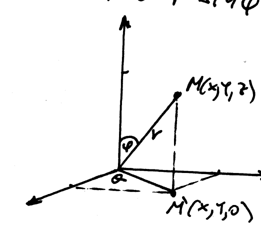
uvodimo smjenu

$x=r\sin\varphi\cos\theta$

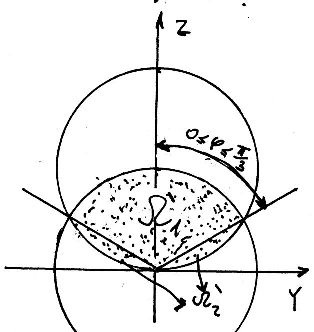
$y=r\sin\varphi\sin\theta$

$z=r\cos\varphi$

$dx\,dy\,dz=r^2\sin\varphi\,dr\,d\varphi\,d\theta$



Napravimo projekciju oblasti na yoz ravan.



$x^2+y^2+z^2\le1$

$(r\sin\varphi\cos\theta)^2+(r\sin\varphi\sin\theta)^2+(r\cos\varphi)^2\le1$

$r^2\le1$

$0\le r\le1$

$x^2+y^2+z^2\le2z$

$r^2\le2r\cos\varphi \quad /:r$

$r\le2\cos\varphi$

$0\le r\le2\cos\varphi$

$0\le\cos\varphi\le1 \quad \forall\varphi$

Može biti $2\cos\varphi < 1$ i $2\cos\varphi > 1$.

1° $2\cos\varphi < 1 \Rightarrow \cos\varphi < \frac{1}{2}$ (pa kako je $\cos\varphi > 0$) $\Rightarrow \varphi \in (\frac{\pi}{3}, \frac{\pi}{2})$

$$\Omega_2 : \begin{cases} 0 \le r \le 2\cos\varphi \\ \frac{\pi}{3} \le \varphi \le \frac{\pi}{2} \\ 0 \le \theta \le 2\pi \end{cases}$$

2° $2\cos\varphi > 1 \Rightarrow \cos\varphi > \frac{1}{2}$ (pa kako je $\cos\varphi \le 1$) $\Rightarrow \varphi \in (0, \frac{\pi}{3})$

$$\Omega_1 : \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \frac{\pi}{3} \\ 0 \le \theta \le 2\pi \end{cases}$$

$\Omega = \Omega_1 \cup \Omega_2$ (Ω_1' i Ω_2' su projekcije oblasti Ω na yOz ravan) (vidi sliku)

$$x^2 + y^2 = r^2 \sin^2\varphi \cos^2\theta + r^2 \sin^2\varphi \sin^2\theta = r^2 \sin^2\varphi$$

$$I = \iiint_{\Omega} \sqrt{x^2+y^2}\, dx\, dy\, dz = \iiint_{\Omega_1} \sqrt{x^2+y^2}\, dx\, dy\, dz + \iiint_{\Omega_2} \sqrt{x^2+y^2}\, dx\, dy\, dz = I_1 + I_2$$

$$I_1 = \iiint_{\Omega_1} \sqrt{r^2 \sin^2\varphi}\, r^2 \sin\varphi\, dr\, d\varphi\, d\theta = \int_0^{2\pi} d\theta \int_0^1 r^3 dr \int_0^{\pi/3} \sin^2\varphi\, d\varphi =$$

$$= \int_0^{2\pi} d\theta \int_0^1 r^3 dr \int_0^{\pi/3} \frac{1}{2}(1 - \cos 2\varphi)\, d\varphi = \frac{1}{2}\int_0^{2\pi} d\theta \int_0^1 r^3 \left(\varphi\Big|_0^{\frac{\pi}{3}} - \frac{1}{2}\sin 2\varphi\Big|_0^{\frac{\pi}{3}}\right) = \ldots = \frac{\pi\sqrt{3}}{32} + \frac{\pi^2}{12}$$

$$I_2 = \int_0^{2\pi} d\theta \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^2\varphi \int_0^{2\cos\varphi} r^3 dr = \ldots = \frac{\pi^2}{12} - \frac{\pi\sqrt{3}}{8}$$

$$I = I_1 + I_2 = \frac{2\pi^2}{12} - \frac{3\pi\sqrt{3}}{32}$$ ← traženo rješenje

Zadaci za vježbu

U zadacima 3547 — 3551 transformisati trojni integral $\iiint_{\Omega} f(x, y, z)\, dx\, dy\, dz$ na cilindrične koordinate ρ, φ, z ($x = \rho\cos\varphi$, $y = \rho\sin\varphi, z = z$), ili na sferne kordinate ρ, θ, φ ($x = \rho\cos\varphi \cdot \sin\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\theta$), a zatim ga svesti na trostruki (sa određenim posebnim granicama integracije).

3547. Ω je oblast u prvom oktantu ograničena cilindrom $x^2 + y^2 = R^2$ i ravnima $z = 0$, $z = 1$, $y = x$ i $y = x\sqrt{3}$.

3548. Ω je oblast ograničena cilindrom $x^2 + y^2 = 2x$, ravni $s = 0$ i paraboloidom $z = x^2 + y^2$.

3549. Ω je deo lopte $x^2 + y^2 + z^2 < R^2$ koji leži u prvom oktantu.

3550. Ω je deo lopte $x^2 + y^2 + z^2 < R^2$ koji leži unutar cilindra $(x^2 + y^2)^2 = R^2(x^2 - y^2)$ $(x > 0)$.

3551. Ω je oblast koja predstavlja zajednički deo dve lopte $x^2 + y^2 + z^2 < R^2$ i $x^2 + y^2 + (z - R)^2 < R^2$.

U zadacima 3552 — 3556 izračunati date integrale prelazeći na cilindrične ili sferne koordinate.

3552. $\int_0^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_0^a dz$.

3553. $\int_0^2 dx \int_0^{\sqrt{2x-x^2}} dy \int_0^a z\sqrt{x^2+y^2}\, dz$.

3554. $\int_{-R}^{-R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_0^{\sqrt{R^2-x^2-y^2}} (x^2 + y^2)\, zd$.

3555. $\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz$.

3556. $\iiint_{\Omega} (x^2 + y^2)\, dx\, dy\, dz$. gde je oblast Ω određena nejednakostima $z \geqslant 0$, $r^2 \leqslant x^2 + y^2 + z^2 \leqslant R^2$.

3557. $\iiint_{\Omega} \frac{dx\, dy\, dz}{\sqrt{x^2 + y^2 + (z-2)^2}}$, gde je Ω—lopta $x^2 + y^2 + z^2 < 1$.

3558. $\iiint_{\Omega} \frac{dx\, dy\, dz}{\sqrt{x^2 + y^2 + (z-2)^2}}$, gde je Ω—cilindar $x^2 + y^2 < 1$, $-1 < z < 1$.

Rješenja

3547. $\int_0^1 dz \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_0^R f(\rho\cos\varphi, \rho\sin\varphi, z)\, \rho\, d\rho$.

3548. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} \rho\, d\rho \int_0^{\rho^2} f(\rho\cos\varphi, \rho\sin\varphi. z)\, dz$.

3549. $\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^R f(\rho\cos\varphi\sin\theta, \rho\sin\varphi\sin\theta\ \rho\cos\theta)\, \rho^2\, d\rho$.

3550. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int_0^{R\sqrt{\cos 2\varphi}} \rho\, d\rho \int_{-\sqrt{R^2-\rho^2}}^{\sqrt{R^2-\rho^2}} f(\rho\cos\varphi, \rho\sin\varphi, z)\, dz$.

Rješenja

3551. $\int_0^{2\pi} d\varphi \int_0^{\frac{R\sqrt{3}}{2}} \rho\, d\rho \int_{R-\sqrt{R^2-\rho^2}}^{\sqrt{R^2-\rho^2}} f(\rho\cos\varphi, \rho\sin\varphi, z)\, dz$ ili

$$\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{3}} \sin\theta\, d\theta \int_0^R f(\rho\cos\varphi\sin\theta, \rho\sin\varphi\sin\theta, \rho\cos\theta)\, \rho^2\, d\rho +$$

$$+ \int_0^{2\pi} d\varphi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^{2R\cos\theta} f(\rho\cos\varphi\sin\theta, \rho\sin\varphi\sin\theta, \rho\cos\theta)\, \rho^2\, d\rho.$$

3552. $\frac{\pi a}{2}$. **3553.** $\frac{8}{9}a^2$. **3554.** $\frac{4}{15}\pi R^5$. **3555.** $\frac{\pi}{8}$.

3556. $\frac{4}{25}\pi(R^5 - r^5)$. **3557.** $\frac{2\pi}{3}$.

3558. $\pi\left[3\sqrt{10} + \ln\frac{\sqrt{2}-1}{\sqrt{10}-3} - \sqrt{2} - 8\right]$.

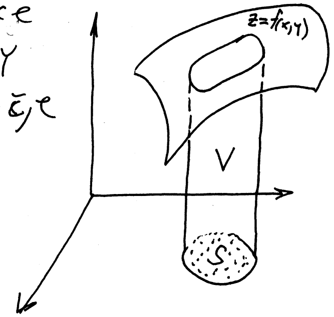
Primjena dvostrukog integrala

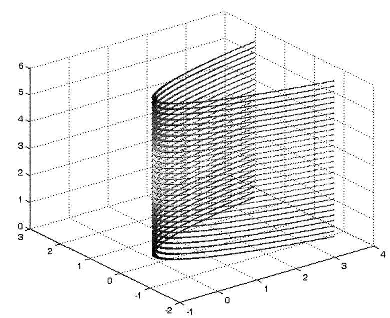
1° Površina zatvorene i ograničene oblasti D

$$P=\iint_D dx\,dy$$

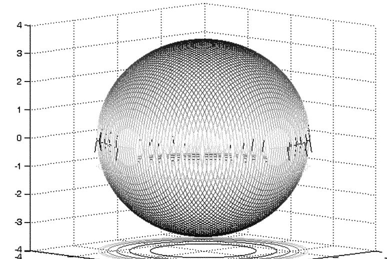
2° Zapremina tijela koje odozgo određuje površ $z=f(x,y)$, odozdo ravan $z=0$ a postranice valjkasta ploha koja na ravni XOY izrezuje omeđeno zatvoreno područje S iznosi

$$V=\iint_S f(x,y)\,dx\,dy$$

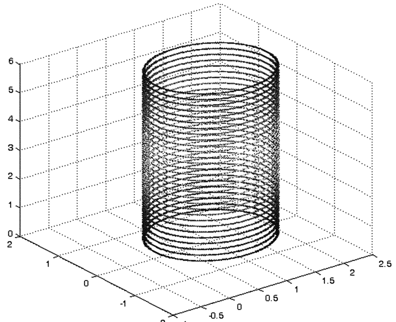




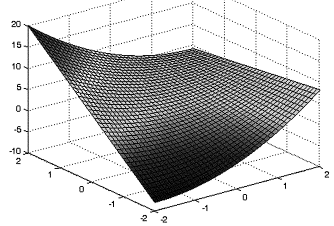
cilindar $x=2y^2$



kugla $x^2+y^2+z^2=12$



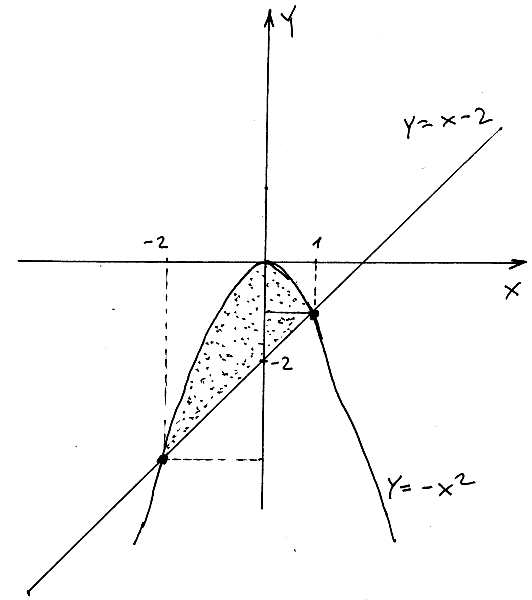
valjak $x^2+y^2=2x$



funkcija $z=x^2-2xy+3y+2$

(#) Naći površinu figure ograničene linijama $y=-x^2$, $x-y-2=0$.

Rj. Nacrtajmo sliku



Pronađimo presječne tačke krive $y=-x^2$ i prave $x-y-2=0$.

$$y=-x^2$$
$$x-y-2=0$$

$$x+x^2-2=0$$
$$x^2+x-2=0$$
$$D=1+8=9 \qquad x_{1,2}=\frac{-1\pm3}{2}$$
$$x_1=-2,\quad x_2=1$$
$$(x-1)(x+2)=0$$
$$x=1 \Rightarrow y=-1$$
$$x=-2 \Rightarrow y=-4$$

I način:

$$P=\int_{-2}^{1}\left(-x^2-(x-2)\right)dx=\int_{-2}^{1}(-x^2-x+2)\,dx=-\frac{1}{3}x^3\Big|_{-2}^{1}-\frac{1}{2}x^2\Big|_{-2}^{1}+2x\Big|_{-2}^{1}=$$

(iznad: $-1+8$, $1-4$, $1+2$)

$$=-\frac{1}{3}\cdot 9+\frac{1}{2}\cdot 3+2\cdot 3=-3+\frac{3}{2}+6=-3-\frac{3}{2}=\frac{9}{2}$$

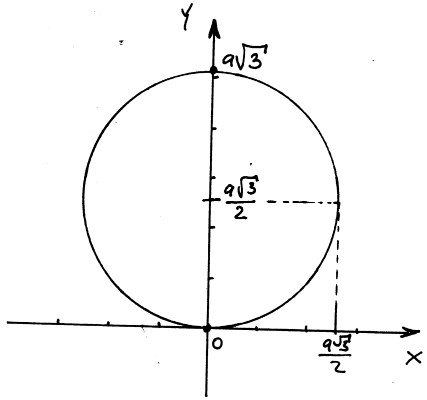
II način:

$$P=\iint_D dx\,dy \quad \text{gdje je} \quad D:\begin{cases}-2\le x\le 1\\ x-2\le y\le -x^2\end{cases}$$

$$P=\iint_D dx\,dy=\int_{-2}^{1}dx\int_{x-2}^{-x^2}dy=\int_{-2}^{1}\left((-x^2)-(x-2)\right)dx=\ldots=\frac{9}{2}$$

Izračunati površinu figure koja je ograničena linijom $x^2+y^2=a\sqrt{3}\,y$.

Rj. $P=\iint\limits_D dx\,dy$



$$x^2+y^2=a\sqrt{3}\,y$$
$$x^2+y^2-a\sqrt{3}\,y=0$$
$$x^2+y^2-2\cdot\frac{a\sqrt{3}}{2}y+\frac{a^2\cdot 3}{4}-\frac{3a^2}{4}=0$$
$$x^2+\left(y-\frac{a\sqrt{3}}{2}\right)^2=\left(\frac{a\sqrt{3}}{2}\right)^2$$

krug s centrom u tački $C\left(0,\frac{a\sqrt{3}}{2}\right)$ poluprečnika $\frac{a\sqrt{3}}{2}$.

Uvodim smjene $x=r\cos\varphi$, $y=\frac{a\sqrt{3}}{2}+r\sin\varphi$, $\quad 0\le r\le\frac{a\sqrt{3}}{2}$, $0\le\varphi\le 2\pi$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi}\end{vmatrix}\qquad \frac{\partial x}{\partial r}=\cos\varphi\quad \frac{\partial x}{\partial \varphi}=-r\sin\varphi\qquad \frac{\partial y}{\partial r}=\sin\varphi\quad \frac{\partial y}{\partial \varphi}=r\cos\varphi$$

$dx\,dy=|J|\,dr\,d\varphi$

$J=r$

$$P=\iint\limits_D dx\,dy=\iint\limits_{D'}|r|\,dr\,d\varphi=\int_0^{2\pi}\left[\int_0^{\frac{a\sqrt{3}}{2}}r\,dr\right]d\varphi=$$

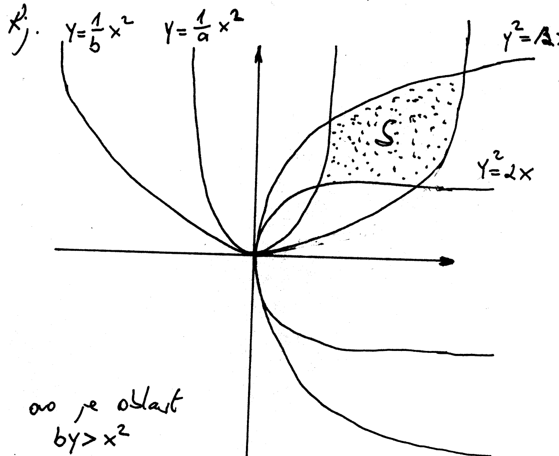
$$=\int_0^{2\pi}\frac{1}{2}r^2\Big|_0^{\frac{a\sqrt{3}}{2}}d\varphi=\frac{a^2\cdot 3}{4}\cdot\frac{1}{2}\,\varphi\Big|_0^{2\pi}=\frac{3a^2}{4}\cdot\pi$$

površina figure koja je ograničena linijom

Izračunati površinu krivoliniskog 4-ugla omeđenog lukovima parabola $x^2=ay$, $x^2=by$, $y^2=\alpha x$ i $y^2=\beta x$ $(0<a<b,\ 0<\alpha<\beta)$.

$P=\iint dx\,dy$

Rj.



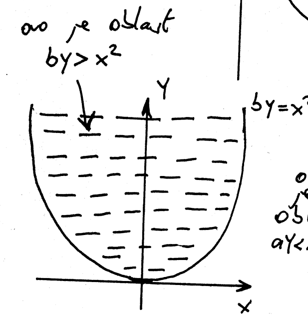
$x^2=ay$, $\quad x^2=by$

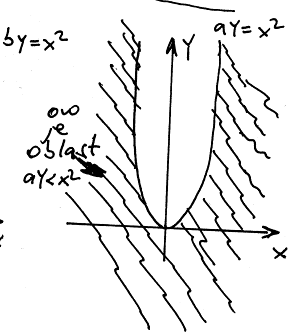
$y=\frac{1}{a}x^2$, $\quad y=\frac{1}{b}x^2$

$a<b$, $\quad \frac{1}{a}>\frac{1}{b}$, $\quad \frac{1}{a}x^2>\frac{1}{b}x^2$

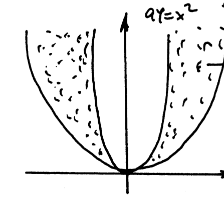
Na klasičan način površinu $\iint\limits_S dx\,dy$ je teško izračunati.

Primjetimo sljedeće:





$\Rightarrow$



oblast $ay\le x^2\le by$, $\quad a\le\frac{x^2}{y}\le b$

Slično $y^2\ge\alpha x$ i $y^2\le\beta x$ imamo $\alpha x\le y^2\le\beta x$, $\quad \alpha\le\frac{y^2}{x}\le\beta$

Vidimo da možemo uvesti smjene $u=\frac{x^2}{y}$ i $v=\frac{y^2}{x}$ gdje

$a\le u\le b$, $\alpha\le v\le\beta$

$u=\frac{x^2}{y}$, $v=\frac{y^2}{x}$, $y=\frac{x^2}{u}$, $x=\frac{y^2}{v}$ $\Rightarrow$ $x=\frac{\left(\frac{x^2}{u}\right)^2}{v}=\frac{x^4}{u^2v}$ $\Rightarrow$ $x^3=u^2v$, $x=\sqrt[3]{u^2v}$

$$y=\frac{x^2}{u}=\frac{\sqrt[3]{(u^2v)^2}}{u}=\sqrt[3]{\frac{u^4v^2}{u^3}}=\sqrt[3]{uv^2}\qquad J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\qquad dx\,dy=|J|\,du\,dv$$

$x=u^{\frac{2}{3}}v^{\frac{1}{3}}$, $\quad \frac{\partial x}{\partial u}=\frac{2}{3}u^{-\frac{1}{3}}v^{\frac{1}{3}}$, $\quad \frac{\partial x}{\partial v}=u^{\frac{2}{3}}\frac{1}{3}v^{-\frac{2}{3}}$

$y=u^{\frac{1}{3}}v^{\frac{2}{3}}$, $\quad \frac{\partial y}{\partial u}=\frac{1}{3}u^{-\frac{2}{3}}v^{\frac{2}{3}}$, $\quad \frac{\partial y}{\partial v}=u^{\frac{1}{3}}\frac{2}{3}v^{-\frac{1}{3}}$

$J=\frac{4}{9}-\frac{1}{9}=\frac{3}{9}=\frac{1}{3}$

$$\iint\limits_S dx\,dy=\int_a^b\left[\int_\alpha^\beta\frac{1}{3}\,dv\right]du=\frac{1}{3}\int_a^b v\Big|_\alpha^\beta du=\frac{1}{3}(\beta-\alpha)\,u\Big|_a^b=\frac{1}{3}(b-a)(\beta-\alpha)$$

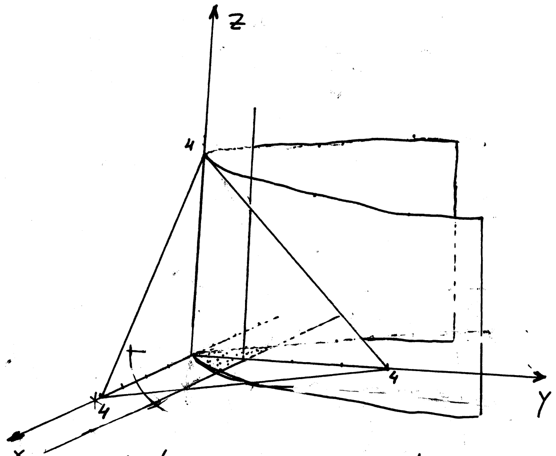
Izračunati zapreminu tijela, ograničeno površinama $y=x^2$, $y=1$, $x+y+z=4$, $z=0$.

Rj. Skicirajmo naše tijelo.

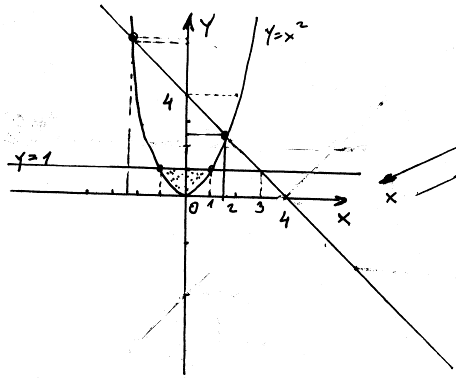
$x+y+z=4$ je ravan ($\frac{x}{4}+\frac{y}{4}+\frac{z}{4}=1$) koja na x, y, i z osi ima odsječke 4.

$y=1$, $z=0$ su ravni

$y=x^2$ je cilindar 



Napravimo ortogonalne projekcije površina na xOy ravan



Nađimo presječnu tačku krive $y=x^2$ i prave $x+y=4$.

$$y=x^2,\quad x+y=4 \qquad y=x^2,\quad y=4-x$$

$$x^2=4-x \qquad x^2+x-4=0 \qquad D=1+16=17 \qquad x_{1,2}=\frac{-1\pm 4{,}12}{2}$$

$$x_1\approx -2{,}56 \Downarrow y_1=2{,}43 \qquad x_2\approx 1{,}56 \Downarrow y_2=6{,}56$$

$$V=\iint_D f(x,y)\,dx\,dy$$ ← zapremina tijela koje je odozgo ograničeno sa f i tijelo ima ortogonalnu projekciju D

U našem slučaju $f(x,y)=4-x-y$ (ovo vidimo sa skice)

$$V=\iint_D (4-x-y)\,dx\,dy \quad \text{gdje je } D:\begin{cases}-1\le x\le 1\\ x^2\le y\le 1\end{cases} \quad \text{ili } D:\begin{cases}0\le y\le 1\\ -\sqrt{y}\le x\le\sqrt{y}\end{cases}$$

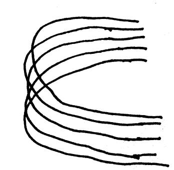
$$V=\int_{-1}^{1}dx\int_{x^2}^{1}(4-x-y)\,dy=\int_{-1}^{1}\Big(\underbrace{4y\Big|_{x^2}^{1}}_{4(1-x^2)}-\underbrace{xy\Big|_{x^2}^{1}}_{-x(1-x^2)}-\underbrace{\tfrac{1}{2}y^2\Big|_{x^2}^{1}}_{-\frac{1}{2}(1-x^4)}\Big)dx=$$

$$=\int_{-1}^{1}\Big(4-4x^2-x+x^3-\tfrac{1}{2}+\tfrac{1}{2}x^4\Big)dx=\int_{-1}^{1}\Big(x^3-4x^2+\tfrac{1}{2}x^4-x+\tfrac{7}{2}\Big)=\dots=-\frac{8}{3}+\frac{1}{5}+7=\frac{68}{15}$$

traženo rješenje

Izračunati zapreminu tijela koje je ograničeno površinama $x=2y^2$, $x+2y+z=4$ i $z=0$.

Rj. $x=2y^2$ cilindar u prostoru



Pronađimo projekciju površina na xOy ravan:

$$x=2y^2,\quad x+2y=4$$

$$2y^2+2y=4 \quad /:2$$

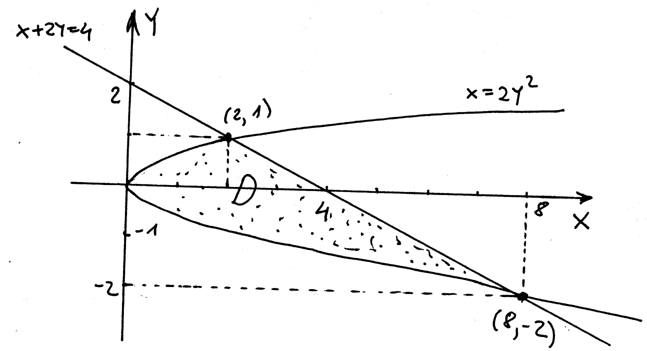
$$y^2+y-2=0$$

$$(y-1)(y+2)=0$$

$$y_1=1 \Rightarrow x_1=2 \qquad y_2=-2 \Rightarrow x_2=8$$

$$x+2y=4 \quad /:4 \qquad \frac{x}{4}+\frac{y}{2}=1$$

Nacrtajmo sliku



$$D:\begin{cases}-2\le y\le 1\\ 2y^2\le x\le 4-2y\end{cases}$$

$$x+2y+z=4 \qquad z=4-x-2y$$

$$V=\iint_D(4-x-2y)\,dx\,dy$$

$$V=\int_{-2}^{1}\Big[\int_{2y^2}^{4-2y}(4-x-2y)\,dx\Big]dy=\int_{-2}^{1}\Big[4x\Big|_{2y^2}^{4-2y}-\tfrac{1}{2}x^2\Big|_{2y^2}^{4-2y}-2y\cdot x\Big|_{2y^2}^{4-2y}\Big]dy=$$

$$=\int_{-2}^{1}\Big[\underbrace{4(4-2y-2y^2)}_{16-8y-8y^2}-\tfrac{1}{2}\underbrace{\big((4-2y)^2-(2y^2)^2\big)}_{16-16y+4y^2-4y^4}-2y(4-2y-2y^2)\Big]dy$$

$$=\int_{-2}^{1}\Big[16-8y-8y^2-8+8y-2y^2+2y^4-8y+4y^2+4y^3\Big]dy=\int_{-2}^{1}(2y^4-6y^2+4y^3-8y+8)\,dy$$

$$=\frac{2}{5}y^5\Big|_{-2}^{1}-\frac{6}{3}y^3\Big|_{-2}^{1}+\frac{4}{4}y^4\Big|_{-2}^{1}-\frac{8}{2}y^2\Big|_{-2}^{1}+8y\Big|_{-2}^{1}=\frac{2}{5}\cdot 33-2\cdot 9+1\cdot(-15)-\frac{8}{2}(-3)$$

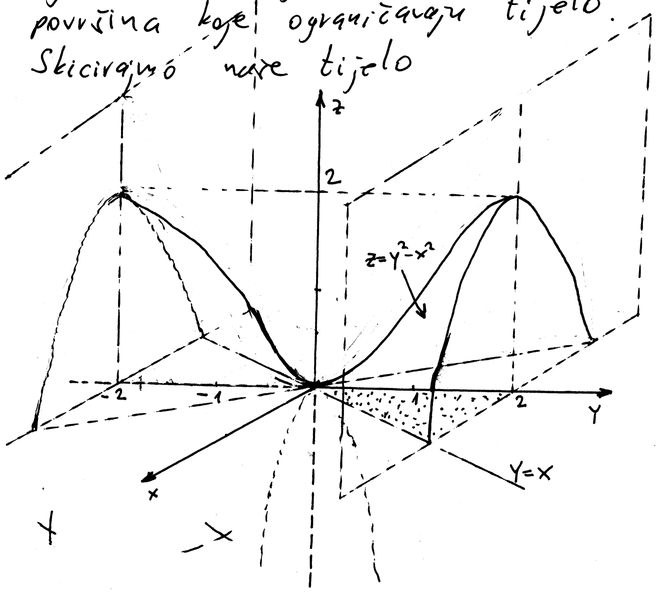
$$+8\cdot 3=\frac{66}{5}-18-15+12+24=\frac{66}{5}+36-33=\frac{66}{5}+\frac{15}{5}=\frac{81}{5}$$

(#) Izračunati zapreminu tijela, koje je ograničeno sa površinama $z = y^2 - x^2$, $z = 0$, $y = \pm 2$.

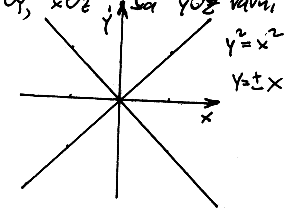
Rj. Zapremina tijela se može računati pomoću dvostrukog ili pomoću trostrukog integrala. Za ta dva slučaja koristimo sljedeće dvije formule

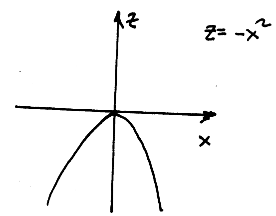
$$V = \iint\limits_D f(x,y)\,dx\,dy \quad , \quad V = \iiint\limits_\Omega dx\,dy\,dz \, .$$

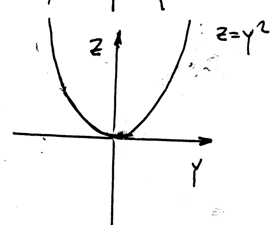
Koju od ove dvije formule je pogodnije koristiti zavisi od jednačina površina koje ograničavaju tijelo.
Skicirajmo naše tijelo



Šta predstavlja jednačina $z = y^2 - x^2$?
Napravimo presjeke $z=y^2-x^2$ sa xOy, xOz i sa yOz ravni







$z(-x,-y) = (-y)^2 - (-x)^2 = y^2 - x^2 = z(x,y)$
$\Rightarrow$ tijelo je simetrično u odnosu na koordinatni početak

$z(x,-y) = (-y)^2 - x^2 = y^2 - x^2$
$\Rightarrow$ tijelo je simetrično u odnosu na xOz osu

$z(-x,y) = y^2 - (-x)^2 = y^2 - x^2 \Rightarrow$ tijelo je simetrično u odnosu na yOz -osu

Sa slike vidimo da možemo izabrati formulu za računanje zapremine $V = \iint\limits_D f(x,y)\,dx\,dy$ i to

$$V = 4\iint\limits_D (y^2 - x^2)\,dx\,dy \quad \text{gdje je} \quad D: \begin{cases} 0 \le y \le 2 \\ 0 \le x \le y \end{cases} \quad \text{(vidi sliku)}$$

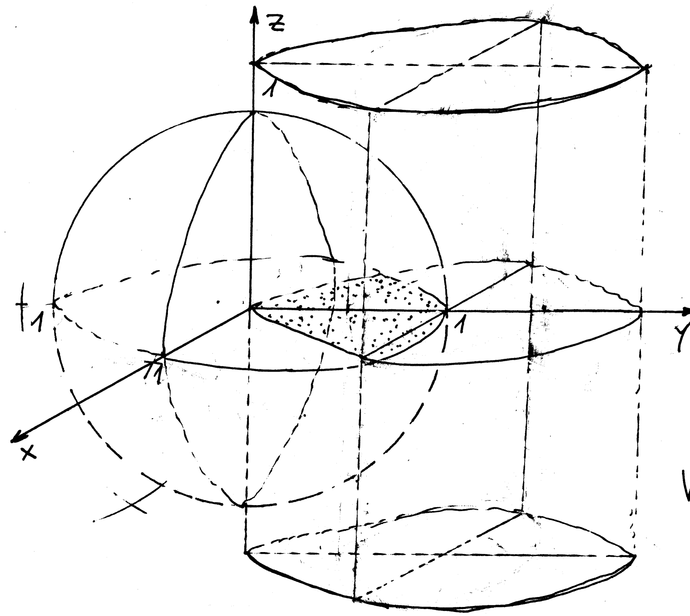
$$V = 4\int_0^2 dy \int_0^y (y^2 - x^2)\,dx = 4\int_0^2 \left(y^2 x \Big|_0^y - \frac{1}{3}x^3 \Big|_0^y \right) dy =$$

$$= 4\int_0^2 \left(y^3 - \frac{1}{3}y^3 \right) dy = 4\int_0^2 \frac{2}{3}y^3\,dy = \frac{8}{3}\cdot\frac{1}{4}y^4\Big|_0^2 = \frac{8}{3}\cdot\frac{1}{4}\cdot 16 = \frac{32}{3}$$

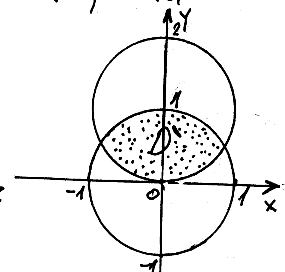
↑ traženo rješenje

Izračunati zapreminu onog dijela lopte $x^2+y^2+z^2=1$ koji se nalazi unutar cilindra $x^2+(y-1)^2=1$.

Rj: Nacrtajmo skicu ove dvije figure u prostoru



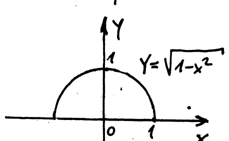
projekcije na xOy osu

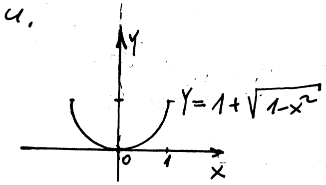


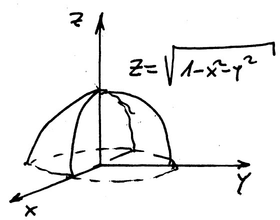
$$V=\iint_D z(x,y)\,dx\,dy$$

zapremina tijela koji je odozgo ograničen sa površi z a čija je projekcija na xOy ravan oblast D

Primjetimo da je presjek cilindra i lopte prvo simetričan u odnosu na xOY osu, a drugo da je simetričan u odnosu na yOz osu.







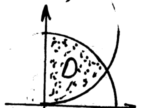
$$\frac{1}{4}V=\int_0^1 dx\int_{1+\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{1-x^2-y^2}\,dy$$

$$\frac{1}{2}V=\iint_{D'} z(x,y)\,dx\,dy,\quad D':\begin{cases}-1\le x\le 1\\ 1+\sqrt{1-x^2}\le y\le\sqrt{1-x^2}\end{cases}$$

uvedimo polarne koordinate

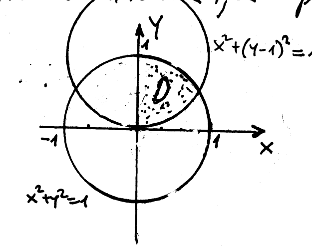
$x=r\cos\varphi$
$y=r\sin\varphi$
$dx\,dy=r\,dr\,d\varphi$

Kako opisati oblast



uz pomoć polarnih koordinata?

Opišimo unutrašnjost presjeka dva kruga pomoću polarnih koordinata.



$x^2+y^2\le 1$
$(r\cos\varphi)^2+(r\sin\varphi)^2\le 1$
$r^2\le 1$
$0\le r\le 1$

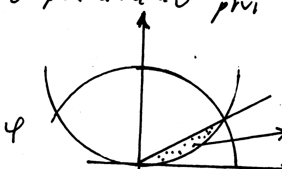
$x^2+(y-1)^2\le 1$
$x^2+y^2-2y+1\le 1$
$x^2+y^2\le 2y$
$r^2\le 2r\sin\varphi \quad /:r$
$r\le 2\sin\varphi$
$0\le r\le 2\sin\varphi$

Kako je $0\le\sin\varphi\le 1$ (ako posmatramo prvi kvadrant) to je moguć i slučaj da je $2\sin\varphi>1$ pa imamo dva slučaja

1° $2\sin\varphi\le 1 \Rightarrow \sin\varphi\le\frac{1}{2}$ (pa ako posmatramo ($\sin\varphi\ge 0$) prvi kvadrant)

$\Rightarrow \varphi\in(0,\frac{\pi}{6})$

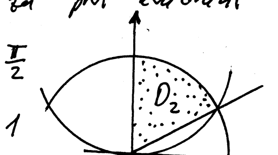
$$D_1:\begin{cases}0\le\varphi\le\frac{\pi}{6}\\ 0\le r\le 2\sin\varphi\end{cases}$$



2° $2\sin\varphi\ge 1 \Rightarrow \sin\varphi\ge\frac{1}{2}$ (pa za prvi kvadrant $\sin\varphi\le 1$)

$\Rightarrow \varphi\in(\frac{\pi}{6},\frac{\pi}{2})$

$$D_2:\begin{cases}\frac{\pi}{6}\le\varphi\le\frac{\pi}{2}\\ 0\le r\le 1\end{cases}$$



$D=D_1\cup D_2$

$$\frac{1}{4}V=\iint_D\sqrt{1-r^2}\,r\,dr\,d\varphi=\iint_{D_1} r\sqrt{1-r^2}\,dr\,d\varphi+\iint_{D_2} r\sqrt{1-r^2}\,dr\,d\varphi$$

$$\iint_{D_1} r\sqrt{1-r^2}\,dr\,d\varphi=\int_0^{\frac{\pi}{6}}d\varphi\int_0^{2\sin\varphi} r\sqrt{1-r^2}\,dr=\left|\begin{array}{c}d(1-r^2)=\\=-2r\,dr\end{array}\right|=\int_0^{\frac{\pi}{6}}d\varphi\int_0^{2\sin\varphi}\left(-\frac{1}{2}\right)\sqrt{1-r^2}\,d(1-r^2)$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{6}}\frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^{2\sin\varphi}d\varphi=-\frac{1}{2}\cdot\frac{2}{3}\int_0^{\frac{\pi}{6}}\left(\Big(1-\underbrace{4\sin^2\varphi}_{(2\sin\varphi)^2}\Big)^{\frac{3}{2}}-1\right)d\varphi$$

Ovo je eliptički integral i on se ne mora izračunati. Njegova približna vrijednost je $\pi/18$.

$$\iint_{D_2} r\sqrt{1-r^2}\,dr\,d\varphi=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}d\varphi\int_0^1\left(-\frac{1}{2}\right)\sqrt{1-r^2}\,d(1-r^2)=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(-\frac{1}{2}\right)\frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^1 d\varphi=-\frac{1}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(-1)\,d\varphi$$

$$=\frac{1}{3}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\frac{1}{3}\cdot\frac{3\pi-\pi}{6}=\frac{1}{3}\cdot\frac{2\pi}{6}=\frac{\pi}{9}$$

$$\frac{1}{4}V=\frac{\pi}{9}+\frac{\pi}{18}=\frac{3\pi}{18}=\frac{\pi}{6}\qquad V=\frac{4\pi}{6}=\frac{2\pi}{3}$$

tražena zapremina

Izračunati zapreminu tijela ograničenog površima:

104. $z = x^2 + y^2,\ y = x^2,\ x = 1,\ z = 0.$

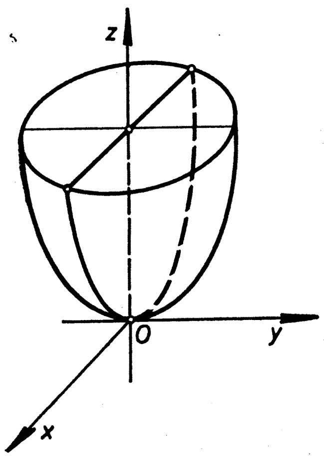
105. $z = xy,\ y = 0,\ x = 0,\ z = 0,\ x^2 + y^2 = r^2.$

Rješenja:

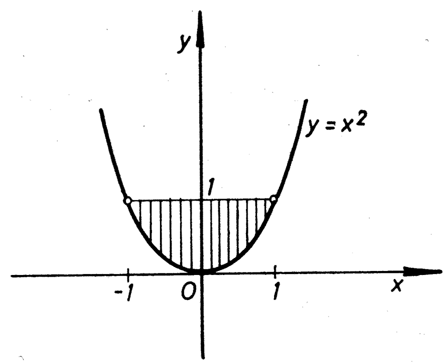
104. Zapremina tijela V ograničenog sa ravni $z = 0$, površi $z = f(x, y)$ $(z \geqslant 0)$ i cilindrom koji izrezuje oblast D (x, y)-ravni, a ima izvodnice paralelne sa z-osom, data je sa

$$V = \iint_D f(x, y)\, dx dy.$$

U ovom slučaju površ $z = f(x, y)$ je paraboloid $z = x^2 + y^2$, (slika 18a) dok je oblast D data na slici 18b.



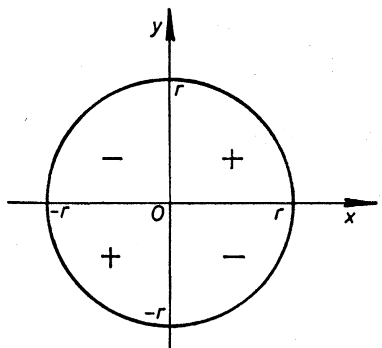
Sl. 18 a



Sl. 18 b

Biće

$$V = \iint_D (x^2 + y^2)\, dx dy = \int_{-1}^{1} dx \int_{x^2}^{1} (x^2 + y^2)\, dy = \frac{88}{105}.$$



Sl. 19

105. Tijelo V se sastoji iz četiri jednaka dijela od kojih su dva ispod ravni $z = 0$ (sl. 19). Biće

$$V = 4 \int_0^r x\, dx \int_0^{\sqrt{r^2 - x^2}} y\, dy =$$

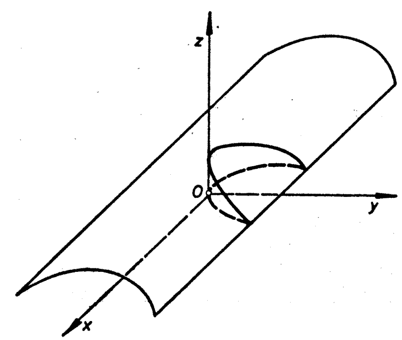
$$= 2 \int_0^r x (r^2 - x^2)\, dx = \frac{r^4}{2}.$$

Izračunati zapreminu tijela ograničenog površima:

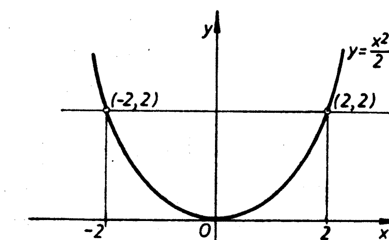
106. $z = 4 - y^2,\ y = \frac{x^3}{2},\ z = 0.$

Rješenja:

106. Površ $z = 4 - y^2$ je parabolični cilindar okomit na ravan yOz, a površ $y = \frac{x^2}{2}$ je parabolični cilindar okomit na ravan xOy (sl. 20). Tijelo V projektuje se na oblast D u ravni $z = 0$ ograničenu parabolom $y = \frac{x^2}{2}$ i presjekom cilindra $z = 4 - y^2$ i ravni $z = 0$ (sl. 21).



Sl. 20



Sl. 21

$$V = \iint_D (4 - y^2)\, dx dy = \int_{-2}^{2} dx \int_{\frac{x^2}{2}}^{2} (4 - y^2)\, dy = 2 \int_0^2 dx \int_{\frac{x^2}{2}}^{2} (4 - y^2)\, dy =$$

$$= 2 \int_0^2 \left(4y - \frac{y^3}{3}\right)\Bigg|_{\frac{x^2}{2}}^{2} dx = 2 \int_0^2 \left(8 - \frac{8}{3} - 2x^2 + \frac{x^6}{24}\right) dx = \frac{256}{21}.$$

(#) Izračunati zapreminu tijela ograničenog površima:

107. $z = 1 - 4x^2 - y^2, \quad z = 0.$

Rješenja:

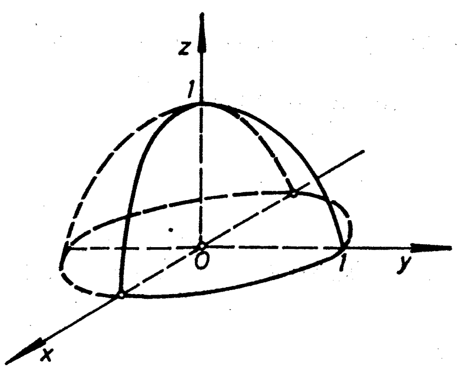
107. Paraboloid $z = 1 - 4x^2 - y^2$ je okrenut nadolje, i siječe se sa ravni $z = 0$ po elipsi $4x^2 + y^2 = 1$ (sl. 22 i sl. 23). Zato je

$$V = \iint_D (1 - 4x^2 - y^2) = \int_{-1/2}^{1/2} dx \int_{-\sqrt{1-4x^2}}^{\sqrt{1-4x^2}} (1 - 4x^2 - y^2)\, dy =$$

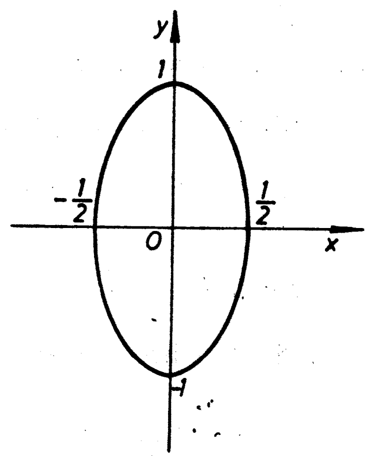
$$= 4 \int_0^{1/2} dx \int_0^{\sqrt{1-4x^2}} (1 - 4x^2 - y^2) dy = \frac{8}{3} \int_0^{1/2} (1 - 4x^2)^{\frac{3}{2}}\, dx.$$

Smjenom $2x = \sin t$ dobija se

$$V = \frac{4}{3} \int_0^{\pi/2} \cos^4 t\, dt = \frac{4}{3} \int_0^{\pi/2} \left(\frac{1 + \cos 2t}{2}\right)^2 dt = \frac{\pi}{4}.$$



Sl. 22



Sl. 23

Zadaci za vježbu

Zapremina tela. I

U zadacima 3559 — 3596 pomoću dvojnih integrala naći zapremine tela ograničenih datim površima (parametre koji ulaze u uslove zadatka smatrati pozitivnim veličinama).

3559. Koordinatnim ravnima, ravnima $x = 4$ i $y = 4$ i obrtnim paraboloidom $z = x^2 + y^2 + 1$.

3560. Koordinatnim ravnima, ravnima $x = a$, $y = b$ i eliptičnim paraboloidom $z = \frac{x^2}{2p} + \frac{y^2}{2q}$.

3561. Ravnima $x = 0$, $y = 0$, $z = 0$ i $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ (piramida).

3562. Ravnima $y = 0$, $z = 0$, $3x + y = 6$, $3x + 2y = 12$ i $x + y + z = 6$.

3563. Obrtnim paraboloidom $z = x^2 + y^2$, koordinatnim ravnima i ravni $x + y = 1$.

3564. Obrtnim paraboloidom $z = x^2 + y^2$ i ravnima $z = 0$, $y = 1$, $y = 2x$ i $y = 6 - x$.

3565. Cilindrima $y = \sqrt{x}$, $y = 2\sqrt{x}$ i ravnima $z = 0$ i $x + z = 6$.

3566. Cilindrom $z = \frac{1}{2} y^2$ i ravnima $x = 0$, $y = 0$, $z = 0$ i $2x + 3y - 12 = 0$.

3567. Cilindrom $z = 9 - y^2$, koordinatnim ravnima i ravni $3x + 4y = 12$ $(y \geqslant 0)$.

3568. Cilindrom $z = 4 - x^2$, koordinatnim ravnima i ravni $2x + y = 4$ $(x \geqslant 0)$.

3569. Cilindrom $2y^2 = x$ i ravnima $\frac{x}{4} + \frac{y}{2} + \frac{z}{4} = 1$ i $z = 0$.

3570. Kružnim cilindrom poluprečnika r, čija se osa poklapa sa ordinatnom osom, koordinatnim ravnima i ravni $\frac{x}{r} + \frac{y}{a} = 1$.

3571. Eliptičnim cilindorom $\frac{x^2}{4} + y^2 = 1$ i ravnima $z = 12 - 3x - 4y$ i $z = 1$.

3572. Cilindrima $x^2 + y^2 = R^2$ i $x^2 + z^2 = R^2$.

3573. Cilindrima $z = 4 - y^2$, $y = \frac{x^2}{2}$ i ravni $z = 0$.

3574. Cilindrima $x^2 + y^2 = R^2$, $z = \frac{x^3}{a^2}$ i ravni $z = 0$ $(x \geqslant 0)$.

3575. Hiperboličnim paraboloidom $z = x^2 - y^2$ i ravnima $z = 0$ i $x = 3$.

3576. Hiperboličnim paraboloidom $z = xy$, cilindrom $y = \sqrt{x}$ i ravnima $x + y = 2$, $y = 0$ i $z = 0$.

3577. Paraboloidom $z = x^2 + y^2$, cilindrom $y = x^2$ i ravnima $y = 1$ i $z = 0$.

3578. Eliptičnim cilindrom $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$ i ravnima $y = \frac{b}{a} x$, $y = 0$ i $z = 0$ $(x > 0)$.

3579. Paraboloidom $z = \frac{a^2 - x^2 - 4y^2}{a}$ i ravni $z = 0$.

3580. Cilindrima $y = e^x$, $y = e^{-x}$, $z = e^2 - y^2$ i ravni $z = 0$.

3581. Cilindrima $y = \ln x$ i $z = \ln^2 x$ i ravnima $z = 0$ i $y + z = 1$.

3582*. Cilindrima $z = \ln x$ i $z = \ln y$ i ravnima $z = 0$ i $x + y = 2e$ $(x > 1)$.

3583. Cilindrima $y = x + \sin x$, $y = x - \sin x$ i $z = \frac{(x+y)^2}{4}$ (parabolički cilindar čije su izvodnice paralelne pravoj $x - y = 0$, $z = 0$) i ravni $z = 0$ $(0 < x \leqslant \pi,\ y > 0)$.

Rješenja

3559. $186\frac{2}{3}$. **3560.** $\frac{ab}{6}\left(\frac{a^2}{p} + \frac{b^2}{q}\right)$.

3561. $\frac{abc}{6}$. **3562.** 12.

3563. $\frac{1}{6}$. **3564.** $78\frac{15}{32}$.

3565. $\frac{48}{5}\sqrt{6}$. **3566.** 16. **3567.** 45.

3568. $13\frac{1}{3}$. **3569.** $16\frac{1}{5}$.

3570. $ar^2\left(\frac{\pi}{4} - \frac{1}{3}\right)$. **3571.** 22π.

3572. $\frac{16}{3}R^3$. **3573.** $12\frac{4}{21}$.

3574. $\frac{4R^5}{15a^2}$. **3575.** 27. **3576.** $\frac{3}{8}$.

3577. $\frac{88}{105}$. **3578.** $\frac{1}{3}abc$.

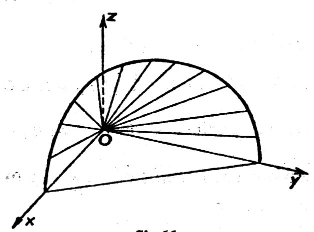
3579. $\frac{\pi a^3}{4}$. **3580.** $2\left(e^2 - \frac{2e^3 + 1}{9}\right)$.

3581. $3e - 8$.

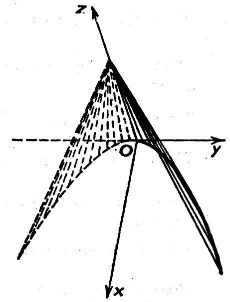
3582*. $4e - e^2 - 1$. Telo je simetrično u odnosu na ravan $y = x$.

3583. $2\left(\pi^2 - \frac{35}{9}\right)$.

3584. Konusnom površinom $z^2=xy$ (sl. 66), cilindrom $\sqrt{x}+\sqrt{y}=1$ i ravni $z=0$.



Sl. 66



Sl. 67

3585. Konusnom površinom $4y^2=x(2-z)$ (parabolični konus, sl. 67) i ravnima $z=0$ i $x+z=2$.

3586. Površinom $z=\cos x\cdot\cos y$ i ravnima $x=0$, $y=0$, $z=0$ i $x+y=\frac{\pi}{2}$.

3587. Cilindrom $x^2+y^2=4$ i ravnima $z=0$ i $z=x+y+10$.

3588. Cilindrom $x^2+y^2=2x$ i ravnima $2x-z=0$ i $4x-z=0$.

3589. Cilindrom $x^2+y^2=R^2$, paraboloidom $Rz=2R^2+x^2+y^2$ i ravni $z=0$.

3590. Cilindrom $x^2+y^2=2ax$, paraboloidom $z=\frac{x^2+y^2}{a}$ i ravni $z=0$.

3591. Sferom $x^2+y^2+z^2=a^2$ i cilindrom $x^2+y^2=ax$ (Vivijanijev problem).

3592. Hiperboličkim paraboloidom $z=\frac{xy}{a}$, cilindrom $x^2+y^2=ax$ i ravni $z=0$ $(x>0,\ y>0)$.

3593. Cilindrima $x^2+y^2=x$ i $x^2+y^2=2x$, paraboloidom $z=x^2+y^2$ i ravnima $x+y=0$, $x-y=0$ i $z=0$.

3594. Cilindrima $x^2+y^2=2x$, $x^2+y^2=2y$ i ravnima $z=x+2y$ i $z=0$.

3595. Konusnom površinom $z^2=xy$ i cilindrom $(x^2+y^2)^2=2xy$ $(x>0,\ y>0,\ z\geqslant 0)$.

3596. Helikoidom („spiralne lestvice") $z=h\operatorname{arctg}\frac{y}{x}$, cilindrom $x^2+y^2=R^2$ i ravnima $x=0$ i $z=0$ $(x>0,\ y\geqslant 0)$.

Površina ravne oblasti

U zadacima 3597 — 3608 pomoću dvojnih integrala naći površine navedenih oblasti.

3597. Oblasti ograničene pravama $x=0$, $y=0$, $x+y=1$.

3598. Oblasti ograničene pravama $y=x$, $y=5x$, $x=1$.

3599. Oblasti ograničene elipsom $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

3600. Oblasti ograničene parabolom $y^2=\frac{b^2}{a}x$ i pravom $y=\frac{b}{a}x$.

3601. Oblasti ograničene parabolama $y=\sqrt{x}$, $y=2\sqrt{x}$ i pravom $x=4$.

3602*. Oblasti ograničene krivom $(x^2+y^2)^2=2ax^3$

3603. Oblasti ograničene krivom $(x^2+y^2)^3=x^4+y^4$.

3604. Oblasti ograničene krivom $(x^2+y^2)^2=2a^2(x^2-y^2)$ (Bernulijeva lemniskata).

3605. Oblasti ograničene petljom krive $x^3+y^3=2xy$ koja leži u prvom kvadrantu.

3606. Oblasti ograničene petljom krive $(x+y)^3=xy$ koja leži u prvom kvadrantu.

3607. Oblasti ograničene petljom krive $(x+y)^5=x^2y^2$ koja leži u prvom kvadrantu.

Rješenja

3584. $\frac{1}{45}$. **3585.** $\frac{16}{9}$. **3586.** $\frac{\pi}{4}$.

3587. 40π. **3588.** 2π.

3589. $\frac{5}{2}\pi R^3$. **3590.** $\frac{3}{2}\pi a^3$.

3591. $\frac{4}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right)$. **3592.** $\frac{a^3}{24}$.

3593. $\frac{15}{8}\left(\frac{3\pi}{8}+1\right)$.

3594. $\frac{3}{2}\left(\frac{\pi}{2}-1\right)$. **3595.** $\frac{\pi\sqrt{2}}{24}$.

3596. $\frac{\pi^2R^2h}{16}$. **3597.** $\frac{1}{2}$.

3598. 2. **3599.** πab.

3600. $\frac{ab}{6}$. **3601.** $\frac{16}{3}$.

3602*. $\frac{5}{8}\pi a^2$; preći na polarne koordinate. **3603.** $\frac{3}{4}\pi$.

3604. $2a^2$. **3605.** $\frac{2}{3}$.

3606. $\frac{1}{60}$. **3607.** $\frac{1}{1260}$.

3608. Oblasti ograničene linijom

1) $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{xy}{c^2}$; 2) $\left(\frac{x^2}{4}+\frac{y^2}{9}\right)^2=\frac{x^2+y^2}{25}$;

Površina površi

3626. Izračunati površinu onog dela ravni $6x+3y+2z=12$ koji leži u prvom oktantu.

3627. Izračunati površinu onog dela površi $z^2=2xy$ koji se projektuje na pravougaonik u ravni $z=0$, ograničen pravama $x=0$. $y=0$, $x=3$, $y=6$.

3628. Naći površinu onog dela konusa $z^2=x^2+y^2$ koji leži između koordinatne ravni Oxy i ravni $z=\sqrt{2}\left(\frac{x}{2}+1\right)$.

U zadacima 3629 — 3639 naći površine naznačenih delova datih površi.

3629. Dela $z^2=x^2+y^2$ isečenog cilindrom $z^2=2py$.

3630*. Dela $y^2+z^2=x^2$, koji leži unutar cilindra $x^2+y^2=R^2$.

3631. Dela $y^2+z^2=x^2$, koji isecaju cilindar $x^2-y^2=a^2$ i ravni $y=b$ i $y=-b$.

3632. Dela $z^2=4x$, koji isecaju cilindar $y^2=4x$ i ravan $x=1$.

3633. Dela $z=xy$, isečenog cilindrom $x^2+y^2=R^2$.

3634. Dela $2z=x^2+y^2$, isečenog cilindrom $x^2+y^2=1$.

3635. Dela $x^2+y^2+z^2=a^2$, isečenog cilindrom $x^2+y^2=R^2$ $(R<a)$.

3636. Dela $x^2+y^2+z^2=R^2$, isečenog cilindrom $x^2+y^2=Rx$.

2637. Dela $x^2+y^2+z^2=R^2$, koga iseca „lemniskatni" cilindar $(x^2+y^2)^2=R^2(x^2-y^2)$.

3638. Dela $z=\frac{x+y}{x^2+y^2}$ koji leži u prvom oktantu i isečen je cilindrima $x^2+y^2=1$ i $x^2+y^2=4$.

3639. Dela $(x\cos\alpha+y\sin\alpha)^2+z^2=a^2$, koji leži u prvom oktantu $\left(\alpha<\frac{\pi}{2}\right)$.

3640*. Izračunati površinu dela zemljine kugle (smatrajući zemlju loptom poluprečnika $R\approx 6400\,km$) ograničenog meridijanima $\varphi=30°$ i $\varphi=60°$, i uporednicima $\theta=45°$ i $\theta=60°$.

3641. Izračunati ukupnu površinu tela ograničenog sferom $x^2+y^2+z^2=3a^2$ i paraboloidom $x^2+y^2=2az$ $(z>0)$.

3642. Ose dva istovetna cilindra poluprečnika R seku se pod pravim uglom; naći površinu onog dela jednog cilindra koji leži u drugom cilindru.

Rješenja

3608*. 1) $\frac{a^2b^2}{2c^2}$; 2) $\frac{39}{25}\pi$; iskoristiti tvrđenje formulisano u zad. 3541.

3626. 14. **3627.** 36.

3628. 8π. **3269.** $2\sqrt{2}\pi p^2$

3630*. $2\pi R^2$. Projicirati površinu na ravan Oyz.

3631. $8\sqrt{2}ab$. **3632.** $\frac{16}{3}(\sqrt{8}-1)$.

3633. $\frac{2\pi}{3}\{(1+R^2)^{\frac{3}{2}}-1\}$.

3634. $\frac{2\pi}{3}(\sqrt{8}-1)$.

3635. $4\pi a(a-\sqrt{a^2-R^2})$.

3636. $2R^2(\pi-2)$.

3637. $2R^2(\pi+4-4\sqrt{2})$.

3638. $\frac{\pi}{4}\{3-\sqrt{2}-\sqrt{3}-\frac{\sqrt{2}}{2}\ln 2+\sqrt{2}\ln(\sqrt{3}+\sqrt{2})\}$.

3639. $\frac{2a^2}{\sin 2\alpha}$.

3640*. $\frac{\pi R^2}{12}(\sqrt{3}-\sqrt{2})\approx 3{,}42\cdot 10^8\ km^2$. Preći na sferne koordinate.

3641. $\frac{16}{3}\pi a^2$. **3642.** $8R^2$.

Primjena trostrukog integrala

a) Zapremina trodimenzionalnog tijela ograničenog oblašću Ω iznosi

$$V=\iiint_{\Omega} dx\,dy\,dz$$

b) Težište $T(x_T, y_T, z_T)$ trodimenzionalnog homogenog tijela ograničenog oblašću Ω tražimo po formulama

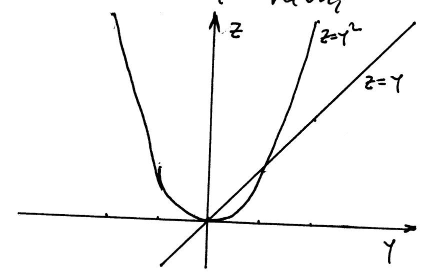
$$x_T = \frac{1}{V}\iiint_{\Omega} x\,dx\,dy\,dz, \qquad y_T = \frac{1}{V}\iiint_{\Omega} y\,dx\,dy\,dz$$

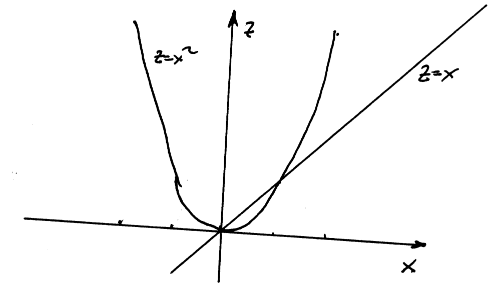
$$z_T = \frac{1}{V}\iiint_{\Omega} z\,dx\,dy\,dz$$

Homogeno tijelo je tijelo kojem je masa jednako raspoređena u svim njegovim dijelovima

(#) Izračunati zapreminu tijela koju ravan $z=x+y$ odsijeca od paraboloida $z=x^2+y^2$.

Rj. Pogledajmo kako izgleda presjek datih površina sa yOz i xOz ravni





Na osnovu ove dvije slike pokušajte skicirati tijelo u prostoru!

$$V=\iiint_{\Omega} dx\,dy\,dz = \iint_{D} dx\,dy \int_{x^2+y^2}^{x+y} dz = \iint_{D} \left(x+y-(x^2+y^2)\right) dx\,dy \overset{(*)}{=}$$

gdje je D ortogonalna projekcija datog tijela na xOy ravan.

Projekciju presjeka tijela određujemo na sljedeći način

$$z=x+y$$
$$z=x^2+y^2$$

$$x+y=x^2+y^2 \Rightarrow x^2-x+y^2-y=0$$

$$x^2-2\cdot x\cdot\frac{1}{2}+\frac{1}{4}+y^2-2\cdot y\cdot\frac{1}{2}+\frac{1}{4}=\frac{1}{2}$$

$$D: \quad \left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2}$$

Ako uvedemo polarne koordinate $x=\frac{1}{2}+r\cos\varphi$, $y=\frac{1}{2}+r\sin\varphi$, $dx\,dy = r\,dr\,d\varphi$

$D \xrightarrow{\text{transformiše}} D'$

$$D' : \begin{cases} 0 \le r \le \frac{1}{\sqrt{2}} \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$\overset{(*)}{=} \iint_D (-1)(x^2 - x + y^2 - y)\, dx\, dy = (-1) \iint_D \left(\left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2 - \frac{1}{2} \right) dx\, dy =$$

Primjetimo da je $x - \frac{1}{2} = r\cos\varphi$

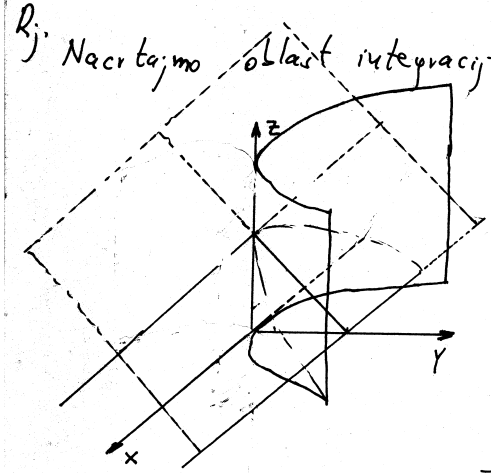
$y - \frac{1}{2} = r \sin\varphi$

$$= (-1) \iint_{D'} \left(r^2 - \frac{1}{2}\right) r\, dr\, d\varphi = (-1) \int_0^{2\pi} d\varphi \int_0^{1/\sqrt{2}} \left(r^3 - \frac{1}{2} r\right) dr = \ldots = \frac{\pi}{8}$$

traženo rješenje

Izračunati zapreminu tijela koje je ograničeno cilindrom $y = 2x^2$ i ravnima $y + z = 8$, $z = 0$.

Rj. Nacrtajmo oblast integracije $\Omega : \begin{cases} y = 2x^2 \\ y + z = 8 \\ z = 0 \end{cases}$

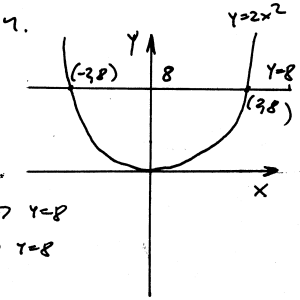


Ravan $y + z = 8$ siječe cilindar. Napravimo projekciju oblasti Ω na xOy ravan.

Nađimo presjek krive $y = 2x^2$ i prave $y = 8$.

$y = 2x^2$
$y = 8$
$x^2 = 4$
$x_1 = -2,\ x_2 = 2$

$x_1 = -2 \Rightarrow y = 8$
$x_2 = 2 \Rightarrow y = 8$



$$\Omega : \begin{cases} -2 \le x \le 2 \\ 2x^2 \le y \le 8 \\ 0 \le z \le 8 - y \end{cases} \qquad V = \iiint_\Omega dx\, dy\, dz$$

$$V = \iiint_\Omega dx\, dy\, dz = \int_{-2}^{2} dx \int_{2x^2}^{8} dy \int_0^{8-y} dz = \int_{-2}^{2} dx \int_{2x^2}^{8} z \Big|_0^{8-y} dy = \int_{-2}^{2} dx \int_{2x^2}^{8} (8 - y)\, dy =$$

$$= \int_{-2}^{2} \left(8y \Big|_{2x^2}^{8} - \frac{1}{2} y^2 \Big|_{2x^2}^{8} \right) dx = \int_{-2}^{2} \left[8(8 - 2x^2) - \frac{1}{2}(8^2 - 4x^4) \right] dx =$$

$$= \int_{-2}^{2} (64 - 16x^2 - 32 + 2x^4)\, dx = \int_{-2}^{2} (2x^4 - 16x^2 + 32)\, dx =$$

$$= 2 \cdot \frac{1}{5} x^5 \Big|_{-2}^{2} - 16 \cdot \frac{1}{3} x^3 \Big|_{-2}^{2} + 32x \Big|_{-2}^{2} = \frac{2}{5} \cdot 64 - \frac{16}{3} \cdot 16 + 32 \cdot 4 =$$

$$= \frac{384 - 1280 + 1920}{15} = \frac{1024}{15}$$

Izračunati zapreminu tijela ograničenog valjkom $x^2+y^2=6x$ i ravnima $x-z=0$, $5x-z=0$.

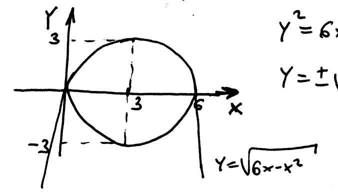
Rj. $V=\iiint\limits_{\Omega} dx\,dy\,dz$

$x^2+y^2=6x$

$x^2-2\cdot x\cdot 3+3^2-3^2+y^2=0$

$(x-3)^2+y^2=3^2$

projekcija valjka na xOy ravan izgleda



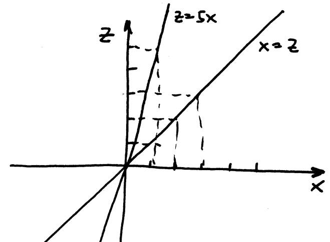
$y^2=6x-x^2$

$y=\pm\sqrt{6x-x^2}$

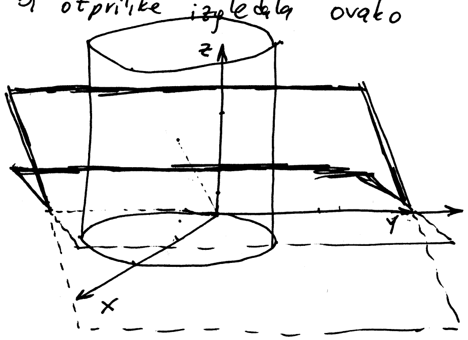
$x-z=0$ $5x-z=0$

$x=z$ $z=5x$

projekcije ravni $x-z=0$ i $5x-z=0$ na xOz ravan izgleda



Skica ovih figura u prostoru bi otprilike izgledala ovako



valjak presječen sa dvije ravni

na klasičan način

$$\Omega:\begin{cases}0<x<6\\ 0<y<\sqrt{6x-x^2}=\sqrt{9-(x-3)^2}\\ x\le z\le 5x\end{cases}$$

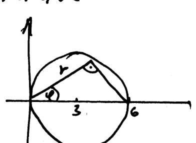
uvedimo cilindrične koordinate

$x=r\cos\varphi$

$y=r\sin\varphi$

$z=z$

$dx\,dy=r\,dr\,d\varphi$



$\cos\varphi=\frac{r}{6}$

Primjetimo da je oblast Ω simetrična u odnosu na xOz ravan

$$V=2\iiint\limits_{\Omega'} r\,dr\,d\varphi\,dz$$

$$\Omega':\begin{cases}0\le\varphi\le\frac{\pi}{2}\\ 0\le r\le 6\cos\varphi\\ r\cos\varphi<z<5r\cos\varphi\end{cases}$$

$$V=2\iiint\limits_{\Omega'} r\,dr\,d\varphi\,dz=2\int_0^{\pi/2}d\varphi\int_0^{6\cos\varphi}r\,dr\int_{r\cos\varphi}^{5r\cos\varphi}dz=8\int_0^{\pi/2}\cos\varphi\,d\varphi\int_0^{6\cos\varphi}r^2\,dr=8\int_0^{\pi/2}\frac{1}{3}r^3\Big|_0^{6\cos\varphi}\cos\varphi\,d\varphi$$

($= 4r\cos\varphi$)

$$=8\cdot\frac{6^3}{3}\int_0^{\pi/2}\cos^4\varphi\,d\varphi=576\int_0^{\pi/2}\left(\frac{1}{2}(1+\cos2\varphi)\right)^2d\varphi=144\int_0^{\pi/2}(1+2\cos2\varphi+\cos^2 2\varphi)\,d\varphi=\ldots=108\pi$$

za vježbu

tražena zapremina

Izračunati zapreminu tijela ograničenog ravninom xOy, valjkom $x^2+y^2=2ax$ i čunjem $x^2+y^2=z^2$.

Rj. Zapremina trodimenzionalnog tijela ograničenog oblašću Ω iznosi $V=\iiint\limits_{\Omega}dx\,dy\,dz$. Pokušajmo skicirati tijelo čiju zapreminu tražimo.

valjak $x^2+y^2=2ax$

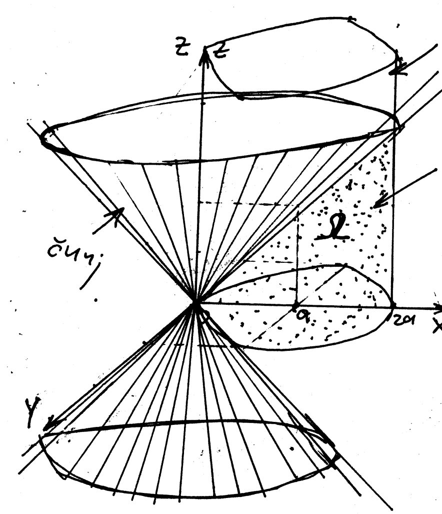
$x^2-2ax+y^2=0$

$x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$

$(x-a)^2+y^2=a^2$

valjak u presjeku sa xOy ravni je krug sa centrom u tački $(a,0)$ poluprečnika a

čunj $x^2+y^2=z^2$ u presjeku sa xOy ravni je tačka, a u presjeku sa yOz ili sa xOz su po dvije prave



Oblast Ω je najlakše projicirati na xOy ravan.

Uvodimo cilindrične koordinate

$x=a+r\cos\varphi$

$y=r\sin\varphi$

$z=z$

$dx\,dy\,dz=r\,dr\,d\varphi\,dz$

$$\Omega':\begin{cases}0\le r\le a\\ 0\le\varphi\le 2\pi\end{cases}$$

$x^2+y^2=z^2$

$z=\pm\sqrt{x^2+y^2}$ čunj

$x^2+y^2=(a+r\cos\varphi)^2+(r\sin\varphi)^2=$

$=a^2+2ar\cos\varphi+r^2\cos^2\varphi+r^2\sin^2\varphi=$

$=a^2+2ar\cos\varphi+r^2$

$$V=\iiint\limits_{\Omega}dx\,dy\,dz=\iiint\limits_{\Omega'}r\,dr\,d\varphi\,dz=\int_0^a dr\int_0^{2\pi}d\varphi\int_0^{\sqrt{a^2+2ar\cos\varphi+r^2}}r\,dz=\ldots$$

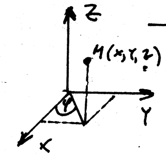
ovo je teško izračunati

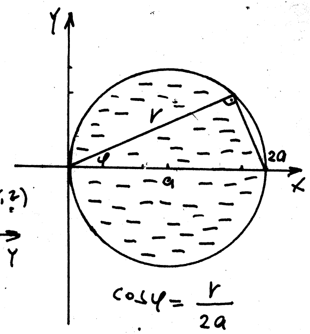
Pokušajmo uvesti drugačije smjene.

$x = r\cos\varphi$ $\quad dx\,dy\,dz = r\,dr\,d\varphi\,dz$

$y = r\sin\varphi$

$z = z$ $\quad x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2$





$$\Omega'':\begin{cases} -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le 2a\cos\varphi \\ 0 \le z \le \sqrt{r^2} \end{cases}$$

$\cos\varphi = \dfrac{r}{2a}$

$r = 2a\cos\varphi$

$$V = \iiint\limits_{\Omega} dx\,dy\,dz = \iiint\limits_{\Omega''} r\,dr\,d\varphi\,dz =$$

$$= \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} dr \int_0^r r\,dz = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} \left(r z\Big|_0^r\right) dr = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} r^2\,dr =$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{3} r^3 \Big|_0^{2a\cos\varphi}\right) d\varphi = \int_{-\pi/2}^{\pi/2} \frac{8}{3} a^3 \cos^3\varphi\,d\varphi = \frac{8}{3} a^3 \int_{-\pi/2}^{\pi/2} \cos^3\varphi\,d\varphi$$

$$\int_{-\pi/2}^{\pi/2} \cos^3\varphi\,d\varphi = \int_{-\pi/2}^{\pi/2} \cos\varphi\cos^2\varphi\,d\varphi = \int_{-\pi/2}^{\pi/2} \cos\varphi(1-\sin^2\varphi)\,d\varphi = \left| \begin{array}{l} \sin\varphi = t \\ \cos\varphi\,d\varphi = dt \\ \varphi = -\frac{\pi}{2} \Rightarrow t = -1 \\ \varphi = \frac{\pi}{2} \Rightarrow t = 1 \end{array} \right|$$

$$= \int_{-1}^{1} (1-t^2)\,dt = t\Big|_{-1}^{1} - \frac{1}{3} t^3 \Big|_{-1}^{1} = 2 - \frac{1}{3}\cdot 2 = \frac{4}{3}$$

$V = \dfrac{32}{9} a^3$ tražena zapremina

II način: $V = \iint\limits_S f(x,y)\,dx\,dy$ uvedimo smjene $x = r\cos\varphi$, $y = r\sin\varphi$, $0 \le \varphi \le 2a\cos\varphi$, $-\frac{\pi}{2} \le r \le \frac{\pi}{2}$

$$V = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} r^2\,dr$$

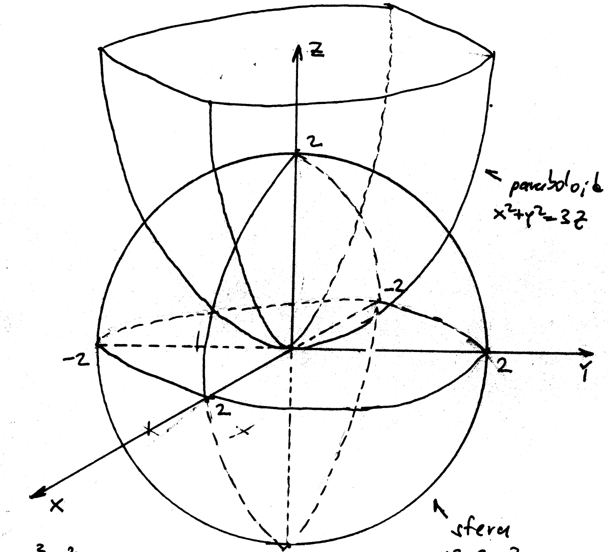
ZAVRŠITI ZA VJEŽBU

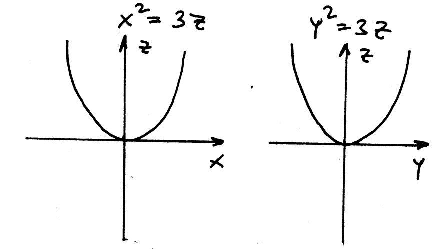
\# Izračunati zapreminu tijela koje je ograničeno površima $x^2+y^2+z^2=4$ i $x^2+y^2=3z$.

Rj. $x^2+y^2+z^2=4$ je sfera sa centrom u $(0,0,0)$ poluprečnika 2

$x^2+y^2=3z$ je paraboloid

Skicirajmo ova dva tijela





$x^2+y^2=4$

$x = \pm\sqrt{4-y^2}$

$y = \pm\sqrt{4-x^2}$

$$V = \iiint\limits_{\Omega} dx\,dy\,dz$$

Primjetimo da je tijelo dobijeno presjekom simetrično na ravni xOz i na yOz. Prema tome

$V = 4\iiint\limits_{\Omega_1} dx\,dy\,dz$ gdje je

Ω_1 oblast u presjeku dva tijela u prvom oktantu

$$\Omega_1:\begin{cases} 0 \le x \le 2 \\ 0 \le y \le \sqrt{4-x^2} \\ 0 \le z \le \frac{1}{3}(x^2+y^2) \end{cases}$$

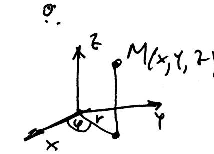
$$V = 4\int_0^2 dx \int_0^{\sqrt{4-x^2}} dy \int_0^{\frac{1}{3}(x^2+y^2)} dz = 4\int_0^2 dx \int_0^{\sqrt{4-x^2}} \frac{1}{3}(x^2-y^2)\,dy$$

$$= \frac{4}{3}\int_0^2 \left(x^2 y\Big|_0^{\sqrt{4-x^2}} + \frac{1}{3} y^3\Big|_0^{\sqrt{4-x^2}}\right) dx = \ldots = \frac{8\pi}{3}$$

komplikovano

II način:

Uvedimo cilindrične koordinate $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$ $\quad dx\,dy\,dz = r\,dr\,d\varphi\,dz$



Oblast $\Omega_1 \xrightarrow{\text{transformacije}} \Omega_1' : \begin{cases} 0 \le r \le 2 \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le z \le \frac{1}{3} r^2 \end{cases}$

$$V = 4\int_0^{\frac{\pi}{2}} d\varphi \int_0^2 dr \int_0^{\frac{1}{3}r^2} r\, dz = 4\int_0^{\frac{\pi}{2}} d\varphi \int_0^2 r \cdot \frac{1}{3} r^2\, dr = \frac{4}{3}\int_0^{\frac{\pi}{2}} \frac{1}{4} r^4 \Big|_0^2 d\varphi = \frac{1}{3}\cdot 16 \cdot \frac{\pi}{2} = \frac{8\pi}{3}$$

$V = \frac{8\pi}{3}$ tražena zapremina

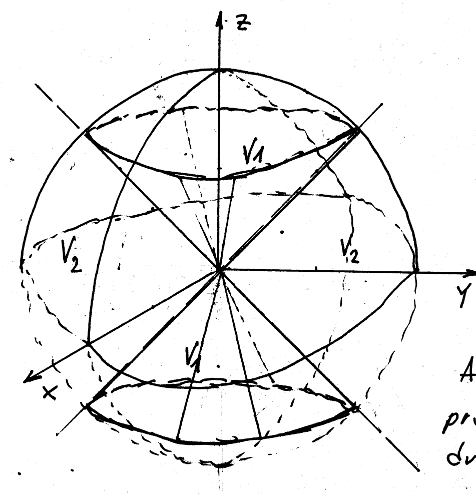
(#) Izračunati zapreminu tijela koje je ograničeno površima $z^2 = x^2 + y^2$, $x^2 + y^2 + z^2 = 4$.

Rj.

$x^2+y^2+z^2=4$ je kugla sa centrom u $(0,0,0)$ poluprečnika $r=2$

$z^2 = x^2+y^2$ je konus

Skicirajmo ove dvije figure u prostoru.



Presjek konusa i kugle daje dva tijela za koje možemo računati zapreminu: prvo tijelo je određeno u presjeku unutrašnjosti konusa i kugle, a drugo tijelo je određeno dijelom lopte van konusa. Ako sa V_1 označimo zapreminu prvog, a sa V_2 zapreminu drugog tijela, imamo da je

$V = V_1 + V_2 = \frac{4}{3} r^3 \pi$ (zapremina kugle)

Kako je $r=2 \Rightarrow V = \frac{4}{3}\cdot 8\pi = \frac{32\pi}{3}$.

$V = \iiint_\Omega dx\,dy\,dz$ – zapremina tijela ograničenog sa oblašću Ω

Uvedimo sferne koordinate

$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

$dx\,dy\,dz = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

$z^2 = x^2 + y^2$

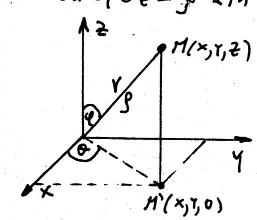
$\rho^2\cos^2\varphi = \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta =$

$= \rho^2\sin^2\varphi(\cos^2\theta + \sin^2\theta) = \rho^2\sin^2\varphi$

$\Rightarrow \cos^2\varphi = \sin^2\varphi \quad /:\sin^2\varphi$

$tg^2\varphi = 1 \Rightarrow tg\varphi = \pm 1$

$x^2+y^2+z^2 = 4 \Rightarrow \rho^2 = 4 \Rightarrow \rho = \pm 2$ tj. $\rho = 2$ (udaljenost tačke)



$\Omega : \begin{cases} z^2 = x^2+y^2 \\ x^2+y^2+z^2 = 4 \end{cases} \xrightarrow{\text{transformacije}} \Omega' : \begin{cases} tg\varphi = \pm 1 \\ \rho = 2 \end{cases}$

Odredimo granice za drugo tijelo $\Omega'_{V_2}: \begin{cases} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ \frac{\pi}{4} \le \varphi \le \frac{3\pi}{4} \end{cases}$

$$V_2 = \iiint\limits_{\Omega'_{V_2}} \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho = \int_0^{2\pi} d\theta \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin\varphi \, d\varphi \int_0^2 \rho^2 d\rho =$$

$$= 2\pi \cdot (-\cos\varphi)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cdot \frac{1}{3}\rho^3\Big|_0^2 = 2\pi\left(-\cos\frac{3\pi}{4} + \cos\frac{\pi}{4}\right) \cdot \frac{8}{3} =$$

$$= 2\pi\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3} = 2\pi\sqrt{2} \cdot \frac{8}{3} = \frac{16\pi\sqrt{2}}{3}$$ traženo rješenje

Zapreminu V_1 sad možemo odrediti na dva načina

I način:

$$V = V_1 + V_2 = \frac{32\pi}{3} \Rightarrow V_1 = \frac{32\pi}{3} - V_2 = \frac{32\pi}{3} - \frac{16\pi\sqrt{2}}{3}$$

$$V_1 = \frac{16\pi}{3}(2 - \sqrt{2})$$ traženo rješenje

II način:

Ako uzmemo u obzir simetričnost date oblasti Ω' u odnosu na xOy-ravan, možemo računati polovinu zapremine V_1 za $z \ge 0$ i tada bi trebalo odabrati sljedeće granice $\Omega'_{V_1}: \begin{cases} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \frac{\pi}{4} \end{cases}$ $\quad V_1 = \iiint\limits_{\Omega'_{V_1}} \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho$

$$\frac{1}{2}V_1 = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} \sin\varphi \, d\varphi \int_0^2 \rho^2 d\rho = 2\pi(-\cos\varphi)\Big|_0^{\frac{\pi}{4}} \cdot \frac{\rho^3}{3}\Big|_0^2 =$$

$$= 2\pi\left(1 - \cos\frac{\pi}{4}\right) \cdot \frac{8}{3} = 2\pi\left(1 - \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3}$$

$$\Rightarrow V_1 = 4\pi\left(1 - \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3} = 4\pi \cdot \frac{2 - \sqrt{2}}{2} \cdot \frac{8}{3} = \frac{16\pi}{3}(2 - \sqrt{2}).$$

(#) Izračunati zapreminu tijela koje je određeno oblašću $\Omega: |x+y+z| + |x-y+z| + |x+y-z| = 1$.

Rj. $V = \iiint_\Omega dx\,dy\,dz$

uvedimo smjenu $\quad u = x+y+z, \quad v = x-y+z, \quad w = x+y-z \qquad dx\,dy\,dz = J\,du\,dv\,dw$ ← Jakobijan

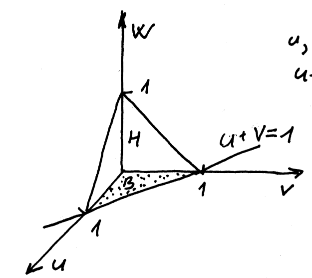
$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \qquad J^{-1} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \begin{matrix} I_v + II_v \\ II_v + III_v \end{matrix}$$

$$= \begin{vmatrix} 2 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & -1 \end{vmatrix} = (-1) \cdot \begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} = (-1)(-4) = 4 \Rightarrow J = \frac{1}{4}$$

pa je $dx\,dy\,dz = \frac{1}{4}du\,dv\,dw$

$\Omega': |u| + |v| + |w| = 1$

$$V = \iiint\limits_{\Omega'} \frac{1}{4} du\,dv\,dw$$



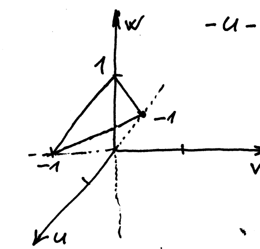
$u, v, w > 0$

$u + v + w = 1$

pored ovoga imamo još 7 slučajeva

npr. $u, v < 0 \quad w > 0$

$-u - v + w = 1$



Vidimo da je dovoljno oblast integrirati u 1. oktantu jer imamo simetričnu oblast po svim oktantima.

$$V = 8 \cdot \frac{1}{4} \iiint\limits_{\Omega''} du\,dv\,dw =$$

$$= 2\int_0^1 du \int_0^{1-u} dv \int_0^{1-u-v} dw = 2\int_0^1 du \int_0^{1-u} w\Big|_0^{1-u-v} dv =$$

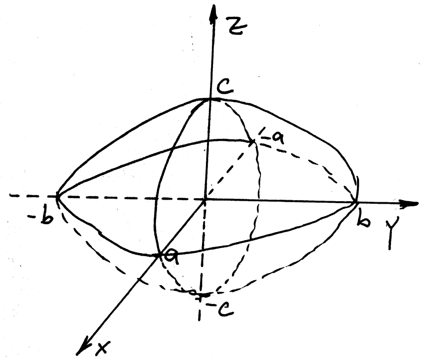
$u + v = 1$, $v = 1 - u$; $u + v + w = 1$, $w = 1 - u - v$

$$= 2\int_0^1 du \int_0^{1-u} (1-u-v)\,dv = 2\int_0^1 \left(v\Big|_0^{1-u} - uv\Big|_0^{1-u} - \frac{1}{2}v^2\Big|_0^{1-u}\right) du = \ldots = 2 \cdot \frac{1}{6} = \frac{1}{3}$$

Na drugi način: $V_1 = \frac{B \cdot H}{3} = \frac{\frac{1 \cdot 1}{2} \cdot H}{3} = \frac{\frac{1}{2} \cdot 1}{3} = \frac{1}{6}, \quad V = 2 \cdot \frac{1}{6} = \frac{1}{3}$ zapremina tijela

Izračunati zapreminu elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Rj.



$$V=\iiint\limits_S dx\,dy\,dz$$

$$S:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

smjena: uopštene sferne koordinate

$x=ar\sin\varphi\cos\theta$ $\quad 0\le r\le 1$

$y=br\sin\varphi\sin\theta$ $\quad 0\le\varphi\le\pi$

$z=cr\cos\varphi$ $\quad 0\le\theta\le 2\pi$

$$dx\,dy\,dz=J\,dr\,d\varphi\,d\theta$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial\theta}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\varphi} & \frac{\partial z}{\partial\theta}\end{vmatrix}=\begin{vmatrix}a\sin\varphi\cos\theta & ar\cos\varphi\cos\theta & -ar\sin\varphi\sin\theta\\ b\sin\varphi\sin\theta & br\cos\varphi\sin\theta & br\sin\varphi\cos\theta\\ c\cos\varphi & -cr\sin\varphi & 0\end{vmatrix}$$

$$=abc\begin{vmatrix}\text{ista determinanta}\\ \text{kao kod standardnih}\\ \text{sfernih koordinata}\end{vmatrix}=abc\,r^2\sin\varphi$$

$$V=\int_0^{\pi}d\varphi\int_0^{1}dr\int_0^{2\pi}abc\,r^2\sin\varphi\,d\theta=\int_0^{\pi}\sin\varphi\,d\varphi\int_0^{1}r^2\,dr\int_0^{2\pi}abc\,d\theta=$$

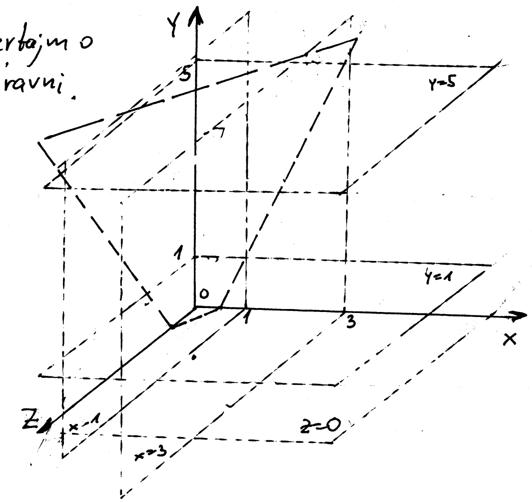
$$=abc\,\theta\Big|_0^{2\pi}\int_0^{\pi}\sin\varphi\,d\varphi\int_0^{1}r^2\,dr=2\pi abc\int_0^{\pi}\sin\varphi\,\frac{1}{3}r^3\Big|_0^1\,d\varphi=$$

$$=\frac{2}{3}\pi abc\int_0^{\pi}\sin\varphi\,d\varphi=\frac{2}{3}\pi abc\left(-\cos\varphi\Big|_0^{\pi}\right)=\frac{2}{3}\pi abc\,(1+1)=\frac{4}{3}\pi abc$$

q.e.d.

Naći zapreminu tijela ograničenog ravnima $x=1$, $x=3$, $y=1$, $y=5$, $2x-y+z-1=0$, $z=0$.

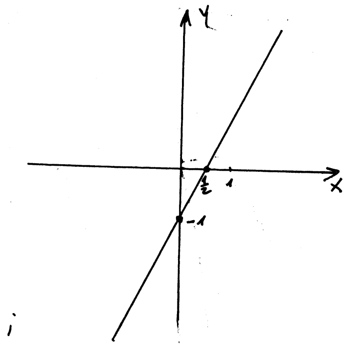
Rj. Nacrtajmo ove ravni.



$2x-y+z-1=0$

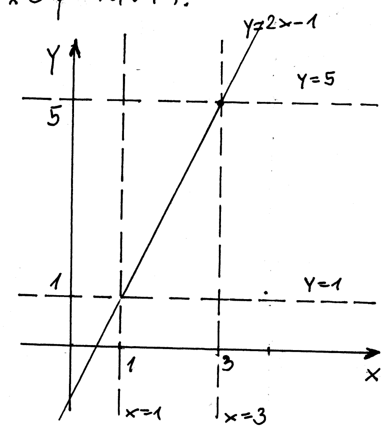
$z=-2x+y+1$

projekcija ove ravni na xOy ravan



$y=2x-1$

Slika u prostoru je komplikovana i sa nje ne možemo pročitati granice. Nacrtajmo projekcije ovih ravni na xOy ravan.



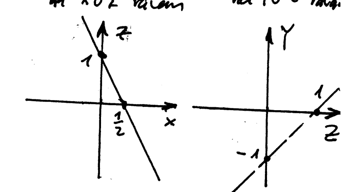
$2x-y-1=0$

$y=2x-1$

$x=3\Rightarrow y=5$

$x=1\Rightarrow y=1$

na xOz ravan / na yOz ravan



$2x+z-1=0$

$z=-2x+1$

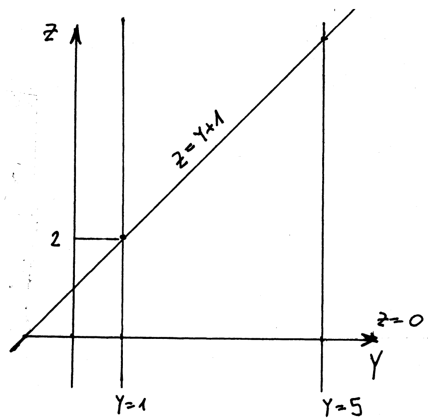
$-y+z-1=0$

$y=z-1$

Sad na osnovu slike u prostoru i projekcija na ravni možemo pročitati granice za tijelo

$$\Omega:\begin{cases}1\le x\le 3\\ 2x-1\le y\le 5\\ 0\le z\le -2x+y+1\end{cases}$$

Da su napisane granice ispravne provjerimo projekcijom ravni na yOz ravan.



$$-y+z-1=0$$
$$z=y+1$$

$$V=\iiint_{\Omega} dx\,dy\,dz =$$

$$=\int_1^3 dx \int_{2x-1}^{5} dy \int_0^{-2x+y+1} dz = \qquad \left(z\Big|_0^{-2x+y+1} \right)$$

$$=\int_1^3 dx \int_{2x-1}^{5} (-2x+y+1)\,dy = \int_1^3 \left((-2x)\cdot y\Big|_{2x-1}^{5} + \frac{1}{2}y^2\Big|_{2x-1}^{5} + y\Big|_{2x-1}^{5} \right) dx =$$

$$=\int_1^3 \left((-2x)(5-(2x-1)) + \frac{1}{2}\left(5^2-(2x-1)^2\right) + 5-(2x-1) \right) dx =$$

$$=\int_1^3 \left((-2x)(6-2x) + \frac{1}{2}\left(25-(4x^2-4x+1)\right) + 6-2x \right) dx =$$

$$=\int_1^3 \left(-12x + 4x^2 + \frac{1}{2}(-4x^2+4x+24) + 6 - 2x \right) dx = \int_1^3 (2x^2-12x+18)\,dx \qquad \left(-2x^2+2x+12\right)$$

$$= \frac{2}{3}x^3\Big|_1^3 - \frac{12}{2}x^2\Big|_1^3 + 18x\Big|_1^3 = \frac{2}{3}\cdot 26 - 6\cdot 8 + 18\cdot 2 = \frac{52}{3} - 12 = \frac{16}{3}$$

(48, 36)

Zapremina tijela ograničenog spomenutim ravnima iznosi $\frac{16}{3}$.

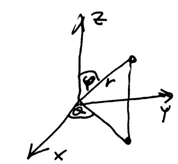
(#) Izračunati zapreminu tijela ograničenog dijelom površi $(x^2+y^2+z^2)^3 = \frac{a^6 z^2}{x^2+y^2}$, $a>0$ u I oktantu.

Rj.

Zapremina tijela ograničenog sa oblasti Ω se računa po formuli $V=\iiint_{\Omega} dx\,dy\,dz$.

Datu površ $(x^2+y^2+z^2)^3 = \frac{a^6 z^2}{x^2+y^2}$ ne možemo skicirati. Uvedimo sferne koordinate

$x = r\sin\varphi\cos\theta$
$y = r\sin\varphi\sin\theta$
$z = r\cos\varphi$



$dx\,dy\,dz = r^2\sin\varphi\,dr\,d\varphi\,d\theta$

$\Omega \xrightarrow{\text{transformiše}} \Omega'$

pa pokušajmo naći granice na osnovu date formule.

$$x^2+y^2+z^2 = r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\varphi\sin^2\theta + r^2\cos^2\varphi = r^2\sin^2\varphi + r^2\cos^2\varphi = r^2$$

$(x^2+y^2+z^2)^3 = (r^2)^3 = r^6$

$z^2 = r^2\cos^2\varphi$

$x^2+y^2 = r^2\sin^2\varphi$

$(x^2+y^2+z^2)^3 = \frac{a^6z^2}{x^2+y^2}$

sad postaje $r^6 = \frac{a^6 r^2\cos^2\varphi}{r^2\sin^2\varphi}$

tj. $r^6 = a^6\,\mathrm{ctg}^2\varphi$

$r = \sqrt[6]{a^6\,\mathrm{ctg}^2\varphi}$

$r = a\sqrt[3]{\mathrm{ctg}\varphi}$.

Na osnovu ove formule i znanja da je tijelo u I oktantu možemo zaključiti da je

$$\Omega': \begin{cases} 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le a\sqrt[3]{\mathrm{ctg}\varphi} \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}$$

$$V = \iiint_{\Omega'} r^2\sin\varphi\,dr\,d\varphi\,d\theta = \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\varphi\,d\varphi \int_0^{a\sqrt[3]{\mathrm{ctg}\varphi}} r^2\,dr = \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\varphi \frac{r^3}{3}\Big|_0^{a\sqrt[3]{\mathrm{ctg}\varphi}} d\varphi$$

$$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \frac{a^3}{3}\sin\varphi\cdot\frac{\cos\varphi}{\sin\varphi}\,d\varphi = \frac{a^3}{3}\int_0^{\pi/2} d\theta \int_0^{\pi/2}\cos\varphi\,d\varphi = \frac{a^3}{3}\cdot\theta\Big|_0^{\frac{\pi}{2}}\cdot\sin\varphi\Big|_0^{\frac{\pi}{2}} = \frac{a^3\pi}{6}$$ tražena zapremina.

Računanje težišta tijela

U slijedećim zadacima izračunajte koordinate težišta tijela (oblasti) Ω ograničenog datim površima!

1. $\Omega: z^2 = xy \wedge x = 5 \wedge y = 5 \wedge z = 0$.

Rješenje: najprije ćemo izračunati zapreminu date oblasti Ω. Očito je $0 \le z \le \sqrt{xy}$, a iz $z^2 = xy$ slijedi $xy \ge 0$, pa je $0 \le x \le 5 \wedge 0 \le y \le 5$. Zato je

$$V = \int_0^5 dx \int_0^5 dy \int_0^{\sqrt{xy}} dz = \int_0^5 \sqrt{x} dx \int_0^5 \sqrt{y} dy = \left(\int_0^5 \sqrt{x} dx \right)^2 = \frac{500}{9}.$$

Dalje imamo da je

$$\overline{x} = \frac{9}{500} \int_0^5 x\, dx \int_0^5 dy \int_0^{\sqrt{xy}} dz = \frac{9}{500} \int_0^5 x\sqrt{x}\, dx \int_0^5 \sqrt{y} dy = \frac{9}{500} \int_0^5 x^{\frac{3}{2}}\, dx \int_0^5 y^{\frac{1}{2}} dy = ... = 3.$$

Očigledno je $\overline{x} = \overline{y}$. Najzad,

$$\overline{z} = \frac{9}{500} \int_0^5 dx \int_0^5 dy \int_0^{\sqrt{xy}} z\, dz = \frac{9}{500} \cdot \frac{1}{2} \int_0^5 x dx \int_0^5 y dy = \frac{9}{1000} \left[\frac{x^2}{2} \bigg|_0^5 \right]^2 = \frac{9}{1000} \cdot \frac{25}{2} \cdot \frac{25}{2} = \frac{45}{32}.$$

Dakle, težište ima koordinate $T\left(3, 3, \frac{45}{32}\right)$.

2. $\Omega: z = 3 - x^2 - y^2, z = 0.$

Rješenje: Uvešćemo cilindrične koordinate. Tada se Ω preslikava u oblast: $\Omega': z = 3 - \rho^2, z = 0.$

U presjeku ove dvije površi se dobija kružnica $\rho^2 = 3 \Rightarrow \rho = \sqrt{3}.$ Zato je $0 \le \varphi \le 2\pi,\ 0 \le \rho \le \sqrt{3},\ 0 \le z \le 3 - \rho^2.$ Odatle slijedi:

$$V = \iiint_{\Omega'} \rho d\varphi\, d\rho\, dz = \int_0^{2\pi} d\varphi \int_0^{\sqrt{3}} \rho d\rho \int_0^{3-\rho^2} dz = 2\pi \int_0^{\sqrt{3}} \rho\left(3 - \rho^2\right) d\rho = 2\pi \int_0^{\sqrt{3}} \left(3\rho - \rho^3\right) d\rho =$$

$$= 2\pi \left(3\frac{\rho^2}{2} - \frac{\rho^4}{4} \right) \bigg|_0^{\sqrt{3}} = 2\pi \left(\frac{9}{2} - \frac{9}{4} \right) = \frac{9\pi}{2}.$$

Sada možemo izračunati koordinate težišta tijela:

$$\overline{x} = \frac{1}{V} \iiint_{\Omega} x\, dx\, dy\, dz = \frac{2}{9\pi} \iiint_{\Omega'} \rho \cos\varphi \cdot \rho\, d\varphi\, d\rho\, dz = \frac{2}{9\pi} \int_0^{2\pi} \cos\varphi\, d\varphi \int_0^{\sqrt{3}} \rho^2 d\rho \int_0^{3-\rho^2} dz = 0,$$

jer je

$\int_0^{2\pi} \cos\varphi\, d\varphi = 0.$ Na isti način dobijamo da je $\overline{y} = 0.$ I najzad,

$$\overline{z} = \frac{1}{V} \iiint_{\Omega'} \rho z\, d\varphi\, d\rho\, dz = \frac{2}{9\pi} \int_0^{2\pi} d\varphi \int_0^{\sqrt{3}} \rho d\rho \int_0^{3-\rho^2} z\, dz = \frac{4\pi}{9\pi} \int_0^{\sqrt{3}} \rho \frac{\left(3 - \rho^2\right)^2}{2} d\rho.$$

U posljednjem integralu zgodno je uzeti smjenu $3 - \rho^2 = t.$ Dobija se dalje da je

$$\overline{z} = \frac{4}{9} \int_3^0 \frac{t^2}{2} \cdot \left(\frac{-1}{2} \right) dt = ... = 1.$$ Znači, $T(0, 0, 1)$.

Napomena: U nekim slučajevima možemo i bez računanja odmah zaključiti da je neka od koordinata težišta jednaka nuli. Radi se o slučajevima kada su jednačine površi koje opisuju oblast Ω simetrične u odnosu na neku od promjenljivih x, y ili z. Tako npr. u posljednjem zadatku, ako označimo $f(x, y, z) = z - \left(3 - x^2 - y^2\right) = x^2 + y^2 - z - 3$, imamo da je $f(x, y, z) = f(-x, y, z)$ i $f(x, y, z) = f(x, -y, z)$, što znači da je funkcija $f(x, y, z)$ simetrična u odnosu na x i u odnosu na y. Zato smo dobili da je $\overline{x} = \overline{y} = 0$.

Zadaci za samostalan rad:

3. $\Omega: z = \frac{y^2}{2},\ x = 0,\ y = 0,\ z = 0,\ 2x + 3y - 12 = 0.$
4. $x^2 + y^2 + z^2 = a^2,\ x^2 + y^2 = ax.$

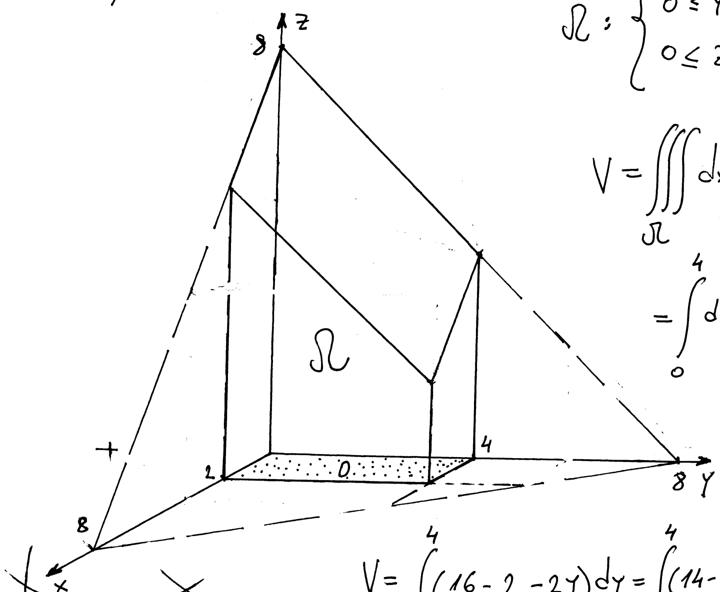
\# Naći težište homogenog tijela ograničenog sa ravnima $x=0$, $y=0$, $z=0$, $x=2$, $y=4$ i $x+y+z=8$ (koso zasječen paralelopiped).

Rj.: Težište $T(x_T, y_T, z_T)$ homogenog tijela ograničenog sa oblašću Ω tražimo po formulama

$$x_T=\frac{1}{V}\iiint_\Omega x\,dx\,dy\,dz,\quad y_T=\frac{1}{V}\iiint_\Omega y\,dx\,dy\,dz,\quad z_T=\frac{1}{V}\iiint_\Omega z\,dx\,dy\,dz$$

gdje je V zapremina tijela Ω.

Skicirajmo dato tijelo



$$\Omega:\begin{cases}0\le x\le 2\\ 0\le y\le 4\\ 0\le z\le 8-x-y\end{cases}$$

$$V=\iiint_\Omega dx\,dy\,dz=\iint_D(8-x-y)\,dx\,dy$$

$$=\int_0^4 dy\int_0^2(8-x-y)\,dx=$$

$$=\int_0^4\left(8x\Big|_0^2-\frac{1}{2}x^2\Big|_0^2-yx\Big|_0^2\right)dy$$

$$V=\int_0^4(16-2-2y)\,dy=\int_0^4(14-2y)\,dy=14y\Big|_0^4-2\cdot\frac{1}{2}y^2\Big|_0^4$$

$$V=14\cdot 4-16=4(14-4)=40$$

$$V=40$$

$$\iiint_\Omega x\,dx\,dy\,dz=\int_0^4 dy\int_0^2 x\,dx\int_0^{8-x-y}dz=\int_0^4 dy\int_0^2(8x-x^2-yx)\,dx=\int_0^4\left(4x^2\Big|_0^2-\frac{1}{3}x^3\Big|_0^2-y\frac{1}{2}x^2\Big|_0^2\right)dy$$

$$=\int_0^4\left(16-\frac{8}{3}-2y\right)dy=\int_0^4\left(\frac{40}{3}-2y\right)dy=\frac{40}{3}y\Big|_0^4-2\cdot\frac{1}{2}y^2\Big|_0^4=\frac{160}{3}-16=\frac{112}{3}$$

$$\iiint_\Omega y\,dx\,dy\,dz=\int_0^2 dx\int_0^4 y\,dy\int_0^{8-x-y}dz=\int_0^2 dx\int_0^4 y(8-x-y)\,dy=\int_0^2 dx\int_0^4(8y-xy-y^2)\,dy=$$

$$=\int_0^2\left(8\frac{1}{2}y^2\Big|_0^4-x\frac{1}{2}y^2\Big|_0^4-\frac{1}{3}y^3\Big|_0^4\right)dx=\int_0^2\left(64-8x-\frac{64}{3}\right)dx=\int_0^2\left(\frac{128}{3}-8x\right)dx=$$

$$=\frac{128}{3}x\Big|_0^2-8\cdot\frac{1}{2}x^2\Big|_0^2=\frac{256}{3}-16=\frac{208}{3}$$

$$\iiint_\Omega z\,dx\,dy\,dz=\ \dots\ (\text{završiti za vježbu})\ =\frac{320}{3}$$

Prema tome, $x_T=\frac{1}{V}\iiint_\Omega x\,dx\,dy\,dz=\frac{1}{40}\cdot\frac{112}{3}=\frac{14}{15}$

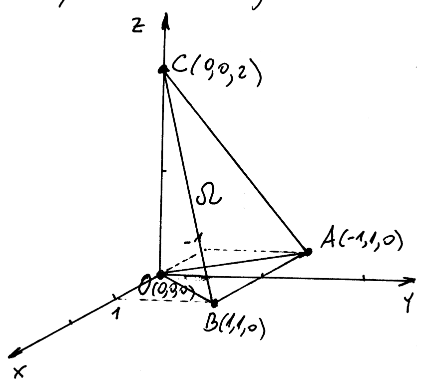
$$y_T=\frac{1}{V}\iiint_\Omega y\,dx\,dy\,dz=\frac{1}{40}\cdot\frac{208}{3}=\frac{25}{15}$$

$$z_T=\frac{1}{V}\iiint_\Omega z\,dx\,dy\,dz=\frac{1}{40}\cdot\frac{320}{3}=\frac{8}{3}$$

Težište homogenog tijela je $T\left(\frac{14}{15},\frac{25}{15},\frac{8}{3}\right)$.

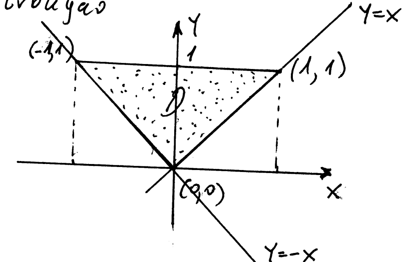
(#) Izračunati pomoću trostrukog integrala zapreminu i težište tetraedra OABC, ako je $O(0,0,0)$, $A(-1,1,0)$, $B(1,1,0)$, $C(0,0,2)$.

Rj: Skicirajmo dato tijelo



$$V=\iiint_{\Omega} dx\,dy\,dz$$

Primjetimo da je projekcija tetraedra na xOy ravan trougao



Odredimo jednačinu ravni kroz tačke A, B i C

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$ jednačina ravni kroz tri tačke

$$\begin{vmatrix} x-(-1) & y-1 & z-0 \\ 1-(-1) & 1-1 & 0-0 \\ 0-(-1) & 0-1 & 2-0 \end{vmatrix}=\begin{vmatrix} x+2 & y-1 & z \\ 2 & 0 & 0 \\ 1 & -1 & 2 \end{vmatrix}=(x+2)\cdot 0-(y-1)\cdot(4-0)+z\cdot(-2-0)$$

$$=-4y+4-2z$$

$-4y+4-2z=0 \quad /:2$

$-2y-z+2=0$ jednačina ravni koja prolazi kroz tačke A, B i C

$$V=\iiint_{\Omega} dx\,dy\,dz=\int_0^1 dy\int_{-y}^{y} dx\int_0^{-2y+2} dz=\int_0^1 dy\int_{-y}^{y} z\Big|_0^{-2y+2} dx=$$

$$=\int_0^1 dy\int_{-y}^{y}(-2y+2)\,dx=\int_0^1\left(-2yx\Big|_{-y}^{y}+2x\Big|_{-y}^{y}\right)dy=\int_0^1(-4y^2+4y)\,dy=-4\cdot\frac{1}{3}y^3\Big|_0^1+4\cdot\frac{1}{2}y^2\Big|_0^1=$$

$y-(-y)$

$$=-\frac{4}{3}+2=\frac{2}{3}$$ traženo rješenje

Zadaci za vježbu

Zapremina tela. II

U zadacima 3609 — 3625 pomoću trojnih integrala izračunati zapremine tela ograničenih datim površinama (parametre koji ulaze u uslove zadatka smatrati pozitivnim veličinama).

3609. Cilindrima $z=4-y^2$ i $z=y^2+2$ i ravnima $x=-1$ i $x=2$.

3610. Paraboloidima $z=x^2+y^2$ i $z=x^2+2y^2$ i ravnima $y=x$, $y=2x$ i $x=1$.

3611. Paraboloidima $z=x^2+y^2$ i $z=2x^2+2y^2$, cilindrom $y=x^2$ i ravni $y=x$.

3612. Cilindrima $z=\ln(x+2)$ i $z=\ln(6-x)$ i ravnima $x=0$, $x+y=2$ i $x-y=2$.

3613*. Paraboloidom $(x-1)^2+y^2=z$ i ravni $2x+z=2$.

3614*. Paraboloidom $z=x^2+y^2$ i ravni $z=x+y$.

3615*. Sferom $x^2+y^2+z^2=4$ i paraboloidom $x^2+y^2=3z$.

3616. Sferom $x^2+y^2+z^2=R^2$ i paraboloidom $x^2+y^2=R(R-2z)$ $(z\geqslant 0)$.

3617. Paraboloidom $z=x^2+y^2$ i konusom $z^2=xy$.

3618. Sferom $x^2+y^2+z^2=4Rz-3R^2$ i konusom $z^2=4(x^2+y^2)$ (misli se na deo loptine zapremine koji leži unutar konusa).

3619*. $(x^2+y^2+z^2)^2=a^3x$.

3620. $(x^2+y^2+z^2)^2=axyz$.

3621. $(x^2+y^2+z^2)^3=a^2z^4$. **3622.** $(x^2+y^2+z^2)^3=\dfrac{a^6z^2}{x^2+y^2}$,

3623. $(x^2+y^2+z^2)^3=a^2(x^2+y^2)^2$.

3624. $(x^2+y^2)^2+z^4=a^3z$.

3625. $x^2+y^2+z^2=1$, $x^2+y^2+z^2=16$, $z^2=x^2+y^2$, $x=0$, $y=0$, $z=0$ $(x>0,\ y>0,\ z\geqslant 0)$.

Težišta homogenih tela

U zadacima 3666 — 3672 naći težišta homogenih tela ograničenih datim površinama.

3666. Ravnima $x=0$, $y=0$, $z=0$, $x=2$, $y=4$ i $x+y+z=8$ (koso zasečeni paralelepiped).

3667. Elipsoidom $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ i koordinatnim ravnima (misli se na deo elipsoida koji leži u prvom oktantu).

3668. Cilindrom $z=\dfrac{y^2}{2}$ i ravnima $x=0$, $y=0$, $z=0$ i $2x+3y-12=0$.

3669. Cilindrima $y=\sqrt{x}$, $y=2\sqrt{x}$ i ravnima $z=0$ i $x+z=0$.

3670. Paraboloidom $z=\dfrac{x^2+y^2}{2a}$ i sferom $x^2+y^2+z^2=3a^2$ $(z>0)$.

3671. Sferom $x^2+y^2+z^2=R^2$ i konusom $z\,\mathrm{tg}\,\alpha=\sqrt{x^2+y^2}$ (loptin isečak).

3672. $(x^2+y^2+z^2)^2=a^3z$.

Rješenja

3666. $\xi=\frac{14}{15}$, $\eta=\frac{26}{15}$, $\zeta=\frac{8}{3}$. **3667.** $\xi=\frac{3}{8}a$, $\eta=\frac{3}{8}b$, $\zeta=\frac{3}{8}c$.

3668. $\xi=\frac{6}{5}$, $\eta=\frac{12}{5}$, $\zeta=\frac{8}{5}$. **3669.** $\xi=\frac{18}{7}$, $\eta=\frac{15}{16}\sqrt{6}$, $\zeta=\frac{12}{7}$.

3670. $\xi=0$, $\eta=0$, $\zeta=\frac{5a}{83}(6\sqrt{3}+5)$.

3671. $\xi=0$, $\eta=0$, $\zeta=\frac{3R}{8}(1+\cos\alpha)$. **3672.** $\xi=0$, $\eta=0$, $\zeta=\frac{9a}{20}$.

Rješenja

3609. 8.

3610. $\frac{7}{12}$. **3611.** $\frac{3}{35}$.

3612. $4(4-3\ln 3)$.

3613*. $\frac{\pi}{2}$. Projekcija tela na ravan xOy je krug.

3614. $\frac{\pi}{8}$. Preneti koordinatni početak u tačku $\left(\frac{1}{2},\frac{1}{2},0\right)$.

3615*. $\frac{19}{6}\pi$ i $\frac{15}{2}\pi$. Preći na cilindrične koordinate.

3616. $\frac{5}{12}\pi R^3$. **3617.** $\frac{\pi}{96}$.

3618. $\frac{92}{75}\pi R^2$.

3619*. $\frac{1}{3}\pi a^3$. Preći na sferne koordinate.

3620. $\frac{a^3}{360}$. **3621.** $\frac{4}{21}\pi a^3$.

3622. $\frac{4}{3}\pi a^3$. **3623.** $\frac{64}{105}\pi a^3$.

3624. $\frac{\pi^2a^3}{6}$. **3625.** $\frac{21(2-\sqrt{2})}{4}\pi$.

Krivoliniski integral prve vrste (po luku)

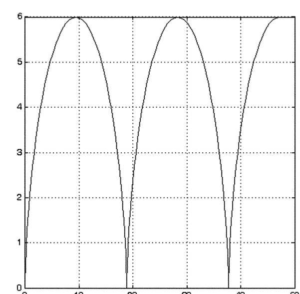
Ako je c kriva data u ravni opisana jednačinom $y=\eta(x)$ gdje je $a\le x\le b$ tada

$$\int_c f(x,y)\,ds = \int_a^b f(x,\eta(x))\underbrace{\sqrt{1+(\eta'(x))^2}\,dx}_{ds}$$

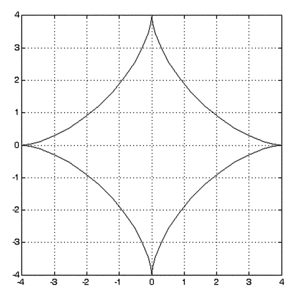
Ako je c kriva opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$ gdje je $t_1\le t\le t_2$ tada $\mu'(t)=\frac{\partial\mu}{\partial t}$

$$\int_c f(x,y)\,ds = \int_{t_1}^{t_2} f(\mu(t),\eta(t))\underbrace{\sqrt{(\mu'(t))^2+(\eta'(t))^2}\,dt}_{ds}$$

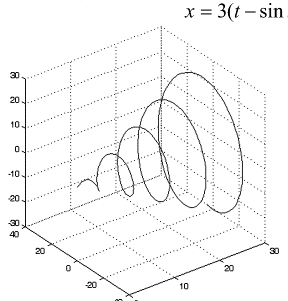
Krivoliniski integrali prve vrste f-ja triju promjenjivih $f(x,y,z)$ uzeti po prostornoj krivoj se računaju analogno. Krivoliniski integral prve vrste NE OVISI O SMJERU PUTA INTEGRACIJE.



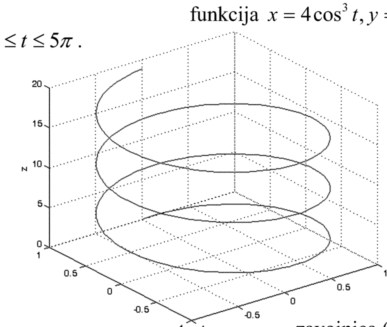
cikloida
$x=3(t-\sin t), y=3(1-\cos t)$, $0\le t\le 5\pi$.



funkcija $x=4\cos^3 t, y=4\sin^3 t$, $0\le t\le 2\pi$.



funkcija $x=t, y=t\cos t, z=t\sin t$, $0\le t\le 30$.



zavojnica (spirala)
$x=\sin t, y=\cos t, z=t$, $0\le t\le 6\pi$.

\# Izračunati krivoliniski integral $I=\int_L(4\sqrt[3]{x}-3\sqrt{y})\,dl$ između tački $E(-1;0)$ i $F(0;1)$

a) po pravoj EF;

b) po liniji astroide $x=\cos^3 t$, $y=\sin^3 t$.

Rj. $I=\int_L(4\sqrt[3]{x}-3\sqrt{y})\,dl$

Ovo je krivoliniski integral prve vrste. Prisjetimo se

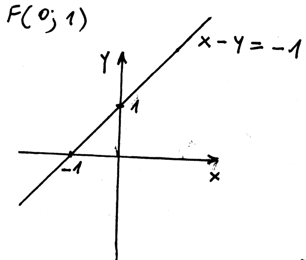
Ako je L kriva u ravni opisana jednačinom $y=\eta(x)$, $a\le x\le b$ tada

$$\int_L f(x,y)\,dl=\int_a^b f(x,\eta(x))\sqrt{1+(\eta'(x))^2}\,dx$$

Ako je L opisana parametarskim jednačinama $\begin{cases}x=\mu(t)\\ y=\alpha(t)\end{cases}$ gdje $t_1\le t\le t_2$

$$\int_L f(x,y)\,dl=\int_{t_1}^{t_2} f(\mu(t),\alpha(t))\sqrt{(\mu'(t))^2+(\alpha'(t))^2}\,dt$$

a) $E(-1;0)$ $F(0;1)$



$-y=-x-1$, $x\in[-1,0]$ tj. $y=x+1$

$y'=1 \Rightarrow dl=\sqrt{1+1^2}\,dx=\sqrt{2}\,dx$

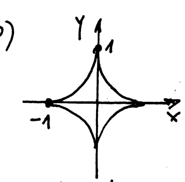
$$I=\int_L(4\sqrt[3]{x}-3\sqrt{y})\,dl=\int_{-1}^{0}(4x^{\frac13}-3(x+1)^{\frac12}\sqrt{2}\,dx$$

$$=4\sqrt{2}\int_{-1}^{0}x^{\frac13}dx-3\sqrt{2}\int_{-1}^{0}(x+1)\,dx=$$

$$=4\sqrt{2}\cdot\frac34 x^{\frac43}\Big|_{-1}^{0}-3\sqrt{2}\int_{-1}^{0}(x+1)^{\frac12}d(x+1)=3\sqrt{2}(0-1)-3\sqrt{2}\cdot\frac23(x+1)^{\frac32}\Big|_{-1}^{0}=-5\sqrt{2}$$

↑ traženo rješenje

b)



astroida

$x=\cos^3 t$, $x'=-3\cos^2 t\sin t$

$y=\sin^3 t$, $y'=3\sin^2 t\cos t$

$$dl=\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t}\,dt$$

$$\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cos^2 t} = 3\sqrt{\cos^2 t\sin^2 t}\,\underbrace{(\cos^2 t+\sin^2 t)}_{=1} =$$

$$= 3|\sin t\cos t|$$

U našem slučaju t uzima vrijednost od $\frac{\pi}{2}$ do π, pa je

$$dl = -3\sin t\cos t\,dt$$

$$I=\int_L\left(4\sqrt[3]{x}-3\sqrt{y}\right)dl = \int_{\frac{\pi}{2}}^{\pi}\left(4\sqrt[3]{\cos^3 t}-3\sqrt{\sin^3 t}\right)(-3\sin t\cos t)\,dt$$

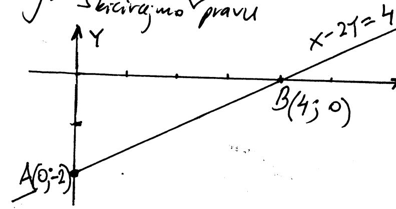
$$= -12\int_{\frac{\pi}{2}}^{\pi}\cos^2 t\sin t\,dt+9\int_{\frac{\pi}{2}}^{\pi}\sin^{\frac52} t\cos t\,dt =$$

$$= +12\int_{\frac{\pi}{2}}^{\pi}\cos^2 t\,d\cos t+9\int_{\frac{\pi}{2}}^{\pi}\sin^{\frac52} t\,d\sin t = 12\cdot\frac{\cos^3 t}{3}\Big|_{\frac{\pi}{2}}^{\pi}+9\cdot\frac{\sin^{\frac72} t}{\frac72}\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= 4\left((-1)^3-0\right)+\frac{18}{7}\left(0-1^{\frac72}\right) = -4-\frac{18}{7} = -\frac{46}{7}$$ traženo rješenje

Izračunati krivolinijski integral $\int_{AB}\frac{dl}{\sqrt{x^2+y^2}}$ po odsječku prave $x-2y=4$ od tačke $A(0;-2)$ do tačke $B(4;0)$.

Rj. Skiciramo datu pravu



Prisjetimo se kako se računa krivolinijski integral prvog tipa, ako je kriva integracije C u ravni opisana formulom $y=\eta(x)$, $a\le x\le b$

$$\int_C f(x,y)\,dl = \int_a^b f(x,\eta(x))\sqrt{1+(\eta'(x))^2}\,dx$$

I način:

$x-2y=4$

$2y=x-4$

$y=\frac12 x-2$

$y'=\frac12$

$$\int_{AB}\frac{dl}{\sqrt{x^2+y^2}} = \int_0^4\frac{\sqrt{1+\frac14}}{\sqrt{x^2+\left(\frac12 x-2\right)^2}}\,dx = \frac{\sqrt5}{2}\int_0^4\frac{dx}{\sqrt{\frac{5x^2}{4}-2x+4}}$$

$$= \frac{\sqrt5}{2}\cdot\frac{1}{\sqrt{\frac54}}\int_0^4\frac{dx}{\sqrt{x^2-\frac85x+\frac{16}{5}}} = \left|\;\begin{aligned}&x^2-\frac85x+\frac{16}{5}=\\&=x^2-2\cdot x\cdot\frac45+\frac{16}{25}-\frac{16}{25}+\frac{16\cdot5}{5\cdot5}\\&=\left(x-\frac45\right)^2+\frac{64}{25}\end{aligned}\right|$$

$$= \int_0^4\frac{d\left(x-\frac45\right)}{\sqrt{\left(x-\frac45\right)^2+\frac{64}{25}}} = \ln\left|x-\frac45+\sqrt{\left(x-\frac45\right)^2+\frac{64}{25}}\right|\Bigg|_0^4 = \ln\left(\frac{16}{5}+\sqrt{\frac{16(16+4)}{25}}\right)-$$

(uz $16+4$: $4\cdot5$; uz $16+64$: $16(1+4)$)

$$-\ln\left(-\frac45+\sqrt{\frac{16+64}{25}}\right) = \ln\left(\frac{16}{5}+\frac{8\sqrt5}{5}\right)-\ln\left(-\frac45+\frac{4\sqrt5}{5}\right)$$

$$= \ln\frac{\frac{16+8\sqrt5}{5}}{\frac{-4+4\sqrt5}{5}} = \ln\frac{4+2\sqrt5}{\sqrt5-1}\cdot\frac{(\sqrt5+1)}{(\sqrt5+1)} = \ln\frac{4+6\sqrt5+10}{5-1} = \ln\frac{7+3\sqrt5}{2}$$ traženo rješenje

II način

$x-2y=4$

$x=2y+4$

$\frac{dx}{dy}=2$

$$\int_{AB}\frac{dl}{\sqrt{x^2+y^2}} = \int_{-2}^0\frac{\sqrt{1+4}}{\sqrt{(2y+4)^2+y^2}}\,dy = \sqrt5\int_{-2}^0\frac{dy}{\sqrt{5y^2+16y+16}} = \dots$$

ZAVRŠITI ZA VJEŽBU

(#) Izračunati krivoliniski integral $\int_L (x-y)\,ds$ po kružnoj liniji $x^2+y^2=ax$.

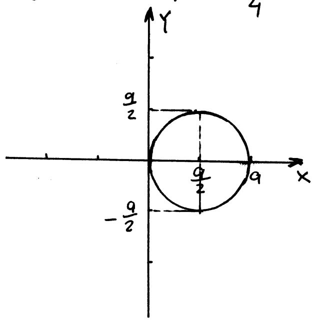
Rj.

$x^2+y^2=ax$

$x^2-ax+y^2=0$

$x^2-2\cdot x\cdot\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}+y^2=0$

$\left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$ krug sa centrom u $C(\frac{a}{2},0)$ poluprečnika $r=\frac{a}{2}$



Kako se računa krivoliniski integral $\int_L f(x,y)\,ds$?

Ako je kriva L data u obliku f-je $y=\eta(x)$ gdje je $a\le x\le b$ tada

$$\int_L f(x,y)\,ds=\int_a^b f(x,\eta(x))\sqrt{1+(y')^2}\,dx.$$

Ako je kriva L data u parametarskom obliku $\begin{cases}x=\mu(t)\\ y=\theta(t)\\ t_1\le t\le t_2\end{cases}$ tada $\int_L f(x,y)\,ds=\int_{t_1}^{t_2} f(\mu(t),\theta(t))\sqrt{\mu'(t)^2+\theta'(t)^2}\,dt$

Prisjetimo se polarnih koordinata $x=r\cos\varphi$, $y=r\sin\varphi$.

Ako pomjerimo centar u x-osi za $\frac{a}{2}$ i fiksiramo r na $\frac{a}{2}$ imamo da je

$$L:\begin{cases}x=\frac{a}{2}+\frac{a}{2}\cos\varphi\\ y=\frac{a}{2}\sin\varphi\\ 0\le\varphi\le 2\pi\end{cases}\qquad x'=-\frac{a}{2}\sin\varphi,\quad y'=\frac{a}{2}\cos\varphi$$

$$(x')^2+(y')^2=\frac{a^2}{4}\sin^2\varphi+\frac{a^2}{4}\cos^2\varphi=\frac{a^2}{4}$$

$$\sqrt{x'^2+y'^2}=\frac{a}{2}$$

$$\int_L (x-y)\,ds=\int_0^{2\pi}\left(\frac{a}{2}+\frac{a}{2}\cos\varphi-\frac{a}{2}\sin\varphi\right)\cdot\frac{a}{2}\,d\varphi=\int_0^{2\pi}\left(\frac{a^2}{4}+\frac{a^2}{4}\cos\varphi-\frac{a^2}{4}\sin\varphi\right)d\varphi=$$

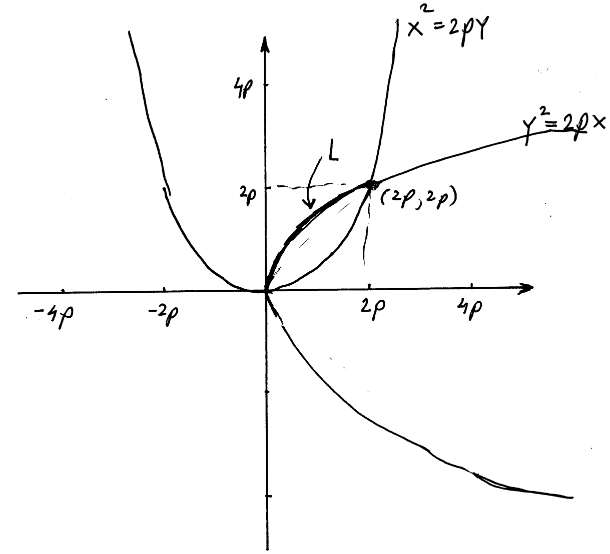
$$=\frac{a^2}{4}\varphi\Big|_0^{2\pi}+\frac{a^2}{4}\sin\varphi\Big|_0^{2\pi}+\frac{a^2}{4}\cos\varphi\Big|_0^{2\pi}=\frac{a^2}{4}\cdot 2\pi=\frac{a^2\pi}{2}$$

traženo rješenje

(#) Izračunati krivoliniski integral $\int_L y\,ds$ pri čemu je L luk parabole $y^2=2px$, koji leži unutar parabole $x^2=2py$.

Rj.

Skicirajmo dvije date parabole



Prisjetimo se kako se računa krivoliniski integral $\int_L f(x,y)\,ds$.

Ako je L kriva u ravni opisana jednačinom $y=\eta(x)$, $a\le x\le b$ se računa

$$\int_L f(x,y)\,ds=\int_a^b f(x,\eta(x))\underbrace{\sqrt{1+(\eta'(x))^2}\,dx}_{ds}$$

U našem slučaju $y=\sqrt{2px}$ gdje je $0\le x\le 2p$

$y=\sqrt{2px}=\sqrt{2p}\cdot\sqrt{x}$ $\qquad (y')^2=\frac{2p}{4x}=\frac{p}{2x}$

$y'=\sqrt{2p}\cdot\frac{1}{2\sqrt{x}}=\frac{p}{\sqrt{2px}}$ $\qquad 1+(y')^2=1+\frac{p}{2x}=\frac{2x+p}{2x}$

$$\int_L y\,ds=\int_0^{2p}\sqrt{2p}\cdot\sqrt{x}\cdot\frac{\sqrt{2x+p}}{\sqrt{2x}}\,dx=\sqrt{2}\cdot\sqrt{p}\cdot\frac{1}{\sqrt{2}}\int_0^{2p}\sqrt{x}\,\frac{\sqrt{2x+p}}{\sqrt{x}}\,dx=\left|\begin{matrix}d(2x+p)=\\=2dx\end{matrix}\right|$$

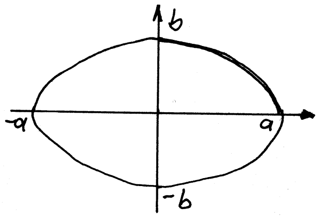
$$=\sqrt{p}\int_0^{2p}\sqrt{2x+p}\cdot\frac{1}{2}\,d(2x+p)=\frac{\sqrt{p}}{2}\cdot\frac{2}{3}\,(2x+p)^{\frac{3}{2}}\Big|_0^{2p}=\frac{p^{\frac{1}{2}}}{3}\left((5p)^{\frac{3}{2}}-p^{\frac{3}{2}}\right)$$

$$=\frac{1}{3}\cdot p^{\frac{1}{2}}\cdot p^{\frac{3}{2}}\left(\sqrt{5}^3-1\right)=\frac{p^2}{3}\left(5\sqrt{5}-1\right)$$

traženo rješenje

Izračunati $\int_c xy\,ds$ gdje je c četvrtina elipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ koja leži u prvom kvadrantu.

Rj: I način:



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$
$$y^2=b^2-\frac{b^2}{a^2}x^2$$
$$y=\pm\sqrt{b^2\left(1-\frac{1}{a^2}x^2\right)}$$
$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$c:\begin{cases} y=\frac{b}{a}\sqrt{a^2-x^2} \\ 0\le x\le a\end{cases}$$

$$y'=\frac{b}{a}\cdot\frac{-2x}{2\sqrt{a^2-x^2}}$$

$$\int_c xy\,ds=\int_0^a x\frac{b}{a}\sqrt{a^2-x^2}\sqrt{1+\left(\frac{b}{a}\cdot\frac{-x}{\sqrt{a^2-x^2}}\right)^2}\,dx$$

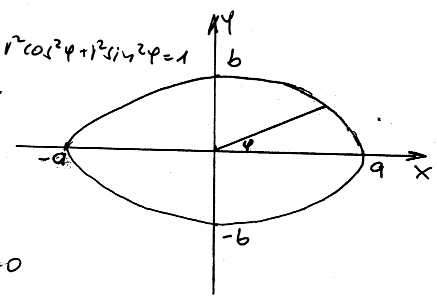
$$=\dots$$

II način

Uvodimo poopštene polarne koordinate

$$x=ar\cos\varphi \qquad x^2=a^2r^2\cos^2\varphi \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=r^2\cos^2\varphi+r^2\sin^2\varphi=1$$
$$y=br\sin\varphi \qquad y^2=b^2r^2\sin^2\varphi$$

za $\varphi=0$ imamo $x=a,\ y=0$
za $\varphi=\frac{\pi}{2}$ imamo $x=0,\ y=b$ $\Big\}\Rightarrow r=1$



Sad elipsu možemo napisati u parametarskom obliku tj. imamo

$$c:\begin{cases} x=a\cos\varphi \\ y=b\sin\varphi \\ 0\le\varphi\le\frac{\pi}{2}\end{cases} \qquad \frac{\partial x}{\partial\varphi}=-a\sin\varphi \qquad \frac{\partial y}{\partial\varphi}=b\cos\varphi$$

$$\int_c f(x,y)\,ds=\int_\alpha^\beta f(\varphi(t),\psi(t))\sqrt{\left(\frac{\partial\varphi}{\partial t}\right)^2+\left(\frac{\partial\psi}{\partial t}\right)^2}\,dt \quad \text{gdje je} \quad \begin{matrix} x=\varphi(t)\\ y=\psi(t)\\ \alpha\le t\le\beta\end{matrix}$$

$$\int_c xy\,ds=\int_0^{\pi/2}(a\cos\varphi)(b\sin\varphi)\sqrt{a^2\underbrace{\sin^2\varphi}_{1-\cos^2\varphi}+b^2\cos^2\varphi}\,d\varphi=ab\int_0^{\frac{\pi}{2}}\cos\varphi\sin\varphi\sqrt{a^2+(b^2-a^2)\cos^2\varphi}\,d\varphi$$

$$=\left|\begin{matrix} a^2+(b^2-a^2)\cos^2\varphi=u & \varphi=0\Rightarrow u=b^2 \\ (b^2-a^2)\,2\cos\varphi(-\sin\varphi)\,d\varphi=du & \varphi=\frac{\pi}{2}\Rightarrow u=a^2\end{matrix}\right|=ab\cdot\frac{-1}{2(b^2-a^2)}\int_{b^2}^{a^2}\sqrt{u}\,du$$

$$=\frac{-ab}{2(b^2-a^2)}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{b^2}^{a^2}=\frac{-ab}{(b-a)(b+a)}\cdot\frac{1}{3}\cdot\frac{(a^3-b^3)}{(a-b)(a^2+ab+b^2)}=\frac{ab}{3(a+b)}(a^2+ab+b^2)$$

Izračunati krivolinijski integral $I=\int_c z\sqrt{x^2+y^2+2z^2}\,dS$

ako je c kriva $x=\frac{r\sqrt{2}}{2}\cos t$, $y=\frac{r\sqrt{2}}{2}\cos t$, $z=r\sin t$, $t\in[0,\pi]$.

Rj: Ako je c kriva opisana parametarskim jednačinama

$$c:\begin{cases} x=\mu(t)\\ y=\eta(t)\\ z=\psi(t)\end{cases},\quad t_1\le t\le t_2 \quad \text{tada}$$

$$\int_c f(x,y,z)\,dS=\int_{t_1}^{t_2} f(\mu(t),\eta(t),\psi(t))\underbrace{\sqrt{(\mu'(t))^2+(\eta'(t))^2+(\psi'(t))^2}\,dt}_{dS}$$

$$x^2+y^2+2z^2=\frac{2r^2}{4}\cos^2t+\frac{2r^2}{4}\cos^2t+2r^2\sin^2t=r^2\cos^2t+2r^2\sin^2t$$

$$x'_t=-\frac{\sqrt{2}}{2}r\sin t,\quad y'_t=-\frac{\sqrt{2}}{2}r\sin t,\quad z'_t=r\cos t$$

$$(x'_t)^2+(y'_t)^2+(z'_t)^2=\frac{1}{2}r^2\sin^2t+\frac{1}{2}r^2\sin^2t+r^2\cos^2t=r^2\sin^2t+r^2\cos^2t$$
$$=r^2(\sin^2t+\cos^2t)=r^2$$

$$\sqrt{(\mu'(t))^2+(\eta'(t))^2+(\psi'(t))^2}=\sqrt{r^2}=r$$

$$I=\int_c z\sqrt{x^2+y^2+2z^2}\,dS=\int_0^\pi r\sin t\sqrt{\underbrace{r^2\cos^2t+2r^2\sin^2t}_{r^2(\cos^2t+2\sin^2t)}}\,r\,dt=$$

$$=r^3\int_0^\pi\sin t\sqrt{\cos^2t+2\underbrace{\sin^2t}_{1-\cos^2t}}\,dt=r^3\int_0^\pi\sin t\sqrt{2-\cos^2t}\,dt=$$

$$=\left|\begin{matrix}\cos t=u \\ -\sin t\,dt=du \\ \sin t\,dt=-du\end{matrix}\quad t\Big|_0^\pi\Rightarrow u\Big|_1^{-1}\right|=r^3\int_{-1}^1\sqrt{2-t^2}\,dt=r^3\int_{-1}^1\frac{2-t^2}{\sqrt{2-t^2}}\,dt$$

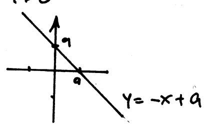
ZA VJEŽBU

$$\dots=r^3\cdot\frac{1}{2}t\sqrt{2-t^2}\Big|_{-1}^1+r^3\int_{-1}^1\frac{dt}{2-t^2}=\dots=\left(1+\frac{\pi}{2}\right)r^3$$

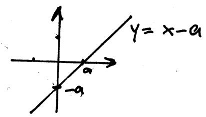
Izračunati integral po krivoj C $\int_C xy\,ds$ gdje je c kvadrat $|x|+|y|=a$, $a>0$.

Rj. Kako nacrtati kvadrat $|x|+|y|=a$?

1° $x>0$, $y>0$: $x+y=a$



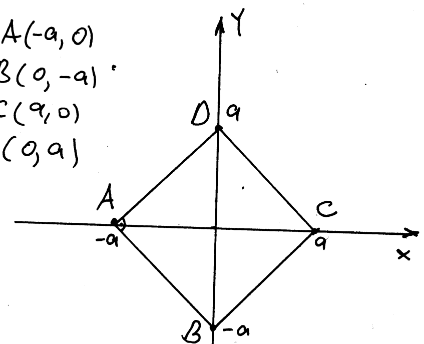
2° $x>0$, $y<0$: $x-y=a$



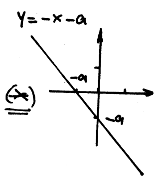
3° $x<0$, $y>0$: $-x+y=a$

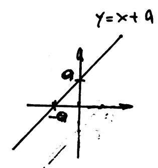
4° $x<0$, $y<0$: $-x-y=a$

$A(-a,0)$
$B(0,-a)$
$C(a,0)$
$D(0,a)$



Kriva po kojoj se integrali mora biti glatka, ako ima čošak razbije se na dijelove.





$$\int_C xy\,ds = \int_{AB} xy\,ds + \int_{BC} xy\,ds + \int_{CD} xy\,ds + \int_{DA} xy\,ds \overset{(*)}{=}$$

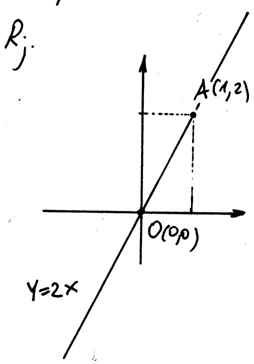
$$\boxed{\int_c f(x,y)\,ds = \int_a^b f(x,\varphi(x))\sqrt{1+(\varphi'(x))^2}\,dx,}$$ gdje je c: $y=\varphi(x)$ kriva $x\in[a,b]$

$$\overset{(*)}{=} \int_{-a}^{0} x\cdot(-x-a)\sqrt{1+(-1)^2}\,dx + \int_0^a x(x-a)\sqrt{1+1^2}\,dx + \int_0^a x(-x+a)\sqrt{1+(-1)^2}\,dx$$

$$+\int_{-a}^{0} x(x+a)\sqrt{1+1^2}\,dx = \sqrt{2}\left(\int_{-a}^{0}(-x^2-ax+x^2+ax)\,dx + \int_0^a (x^2-ax-x^2+ax)\,dx\right) = 0$$

Izračunati integral $\int_c \frac{ds}{\sqrt{x^2+y^2+4}}$ gdje je c duž koja spaja tačke $O(0,0)$ i tačku $A(1,2)$.

Rj.



c: $y=2x$. $y'=2$

$$\boxed{\int_c f(x,y)\,ds = \int_a^b f(x,\varphi(x))\sqrt{1+(\varphi'(x))^2}\,dx}$$ gdje je $y=\varphi(x)$. kriva $x\in[a,b]$

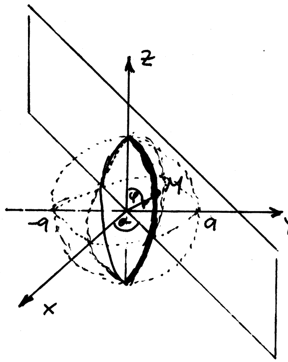
$$\int_c \frac{1}{\sqrt{x^2+y^2+4}}\,ds = \int_0^1 \frac{\sqrt{1+2^2}}{\sqrt{x^2+(2x)^2+4}}\,dx = \sqrt{5}\int_0^1 \frac{dx}{\sqrt{5x^2+4}} =$$

$$=\sqrt{5}\int_0^1 \frac{dx}{\sqrt{4(\frac{5}{4}x^2+1)}} = \frac{\sqrt{5}}{2}\int_0^1 \frac{d(\frac{\sqrt{5}}{2}x)}{\sqrt{(\frac{\sqrt{5}}{2}x)^2+1}}\cdot\frac{2}{\sqrt{5}} = \ln\left|\frac{\sqrt{5}x}{2}+\sqrt{\left(\frac{\sqrt{5}x}{2}\right)^2+1}\right|\Bigg|_0^1$$

$$= \ln\left|\frac{\sqrt{5}}{2}+\sqrt{\frac{5}{4}+1}\right| - \ln 1 = \ln\left|\frac{\sqrt{5}}{2}+\sqrt{\frac{9}{4}}\right| = \ln\frac{\sqrt{5}+3}{2}$$

Izračunati $\int\limits_c \sqrt{2y^2+z^2}\,ds$ gdje je c krug dobijen presjekom sfere $x^2+y^2+z^2=a^2$ i ravni $x=y$.

Rj.



Kako ćemo opisati datu sferu parametarski? (sferne koordinate)

$$x = r\sin\varphi\cos\theta,\quad y = r\sin\varphi\sin\theta,\quad z = r\cos\varphi, \qquad r=a,\quad 0\le\theta\le 2\pi,\quad 0\le\varphi\le\pi$$

Kako da parametarski opišemo krug dobijen presjekom sfere i ravni?

Za pravu $x=y$ znamo da je ugao između ove prave i x-ose $45°$. Prema tome $\theta=45°$, $(r=a)$:

$$c:\begin{cases} x = \frac{\sqrt{2}}{2}a\sin\varphi \\ y = \frac{\sqrt{2}}{2}a\sin\varphi \\ z = a\cos\varphi \\ 0\le\varphi\le 2\pi \end{cases}$$

↑ θ i r su fiksirani

$$2y^2+z^2 = 2\cdot\frac{2}{4}a^2\sin^2\varphi + a^2\cos^2\varphi = a^2$$

Ako je kriva c opisana sa $x=\mu(t)$, $y=\eta(t)$, $z=\xi(t)$, $\alpha<t<\beta$ onda je

$$\int\limits_c f(x,y,z)\,ds = \int_\alpha^\beta f(\mu(t),\eta(t),\xi(t))\sqrt{(\mu'(t))^2+(\eta'(t))^2+(\xi'(t))^2}\,dt$$

$$\frac{\partial x}{\partial\varphi} = \frac{\sqrt{2}}{2}a\cos\varphi$$

$$\frac{\partial y}{\partial\varphi} = \frac{\sqrt{2}}{2}a\cos\varphi$$

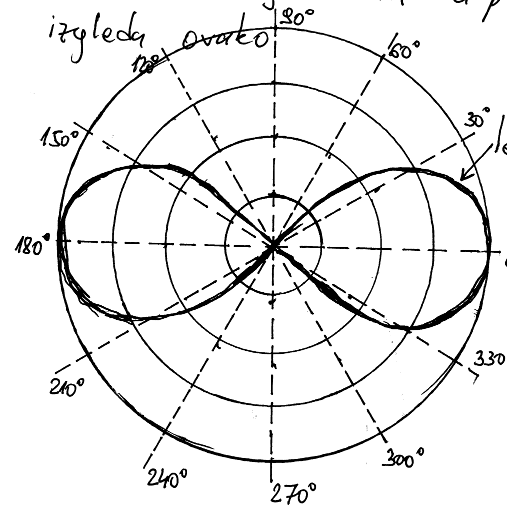
$$\frac{\partial z}{\partial\varphi} = -a\sin\varphi$$

$$\int\limits_c \sqrt{2y^2+z^2}\,ds = \int_0^{2\pi}\sqrt{a^2}\cdot\sqrt{\frac{2}{4}a^2\cos^2\varphi+\frac{2}{4}a^2\cos^2\varphi+a^2\sin^2\varphi}\,dt$$

$$= \int_0^{2\pi} a\cdot\sqrt{a^2(\cos^2\varphi+\sin^2\varphi)}\,dt = a^2\int_0^{2\pi}dt = 2a^2\pi.$$

Izračunati krivolinijski integral prve vrste $\int\limits_c (x+y)\,dS$, ako je c desna latica lemniskate $\rho = a\sqrt{\cos 2\varphi}$.

Rj. Lemniskata $\rho=a\sqrt{\cos 2\varphi}$ u polarnom koordinatnom sistemu izgleda ovako



Data kriva je prikazana u polarnim koordinatama

$$c:\begin{cases}\rho = a\sqrt{\cos 2\varphi}\\ \varphi\in\left[-\frac{\pi}{4},\frac{\pi}{4}\right]\cup\left[\frac{3\pi}{4},\frac{5\pi}{4}\right]\end{cases}$$

Prisjetimo se,

$$\int\limits_c (x+y)\,dS = \int_{t_1}^{t_2}(\eta(t)+\mu(t))\sqrt{(\eta'(t))^2+(\mu'(t))^2}\,dt$$

ako je c data u obliku

$$c:\begin{cases}x=\eta(t)\\ y=\mu(t)\\ t_1\le t\le t_2\end{cases}$$

$$\int\limits_{c'}(x+y)\,dS = \Bigg|$$ kao pomoć uvedimo polarne koordinate $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, za ρ ćemo uzeti $\rho=a\sqrt{\cos 2\varphi}$

Prema tome ← desna latica lemniskate

$$c':\begin{cases}x = a\cos\varphi\sqrt{\cos 2\varphi}\\ y = a\sin\varphi\sqrt{\cos 2\varphi}\\ -\frac{\pi}{4}\le\varphi\le\frac{\pi}{4}\end{cases}$$

↑ zato što posmatramo desnu stranu lemniskate

$$x' = \left(a(-\sin\varphi)\sqrt{\cos 2\varphi} + a\cos\varphi\cdot\frac{1}{2}(\cos 2\varphi)^{-\frac{1}{2}}\cdot(-\sin 2\varphi)\cdot 2\right)d\varphi = a\left(-\sin\varphi\sqrt{\cos 2\varphi} - \cos\varphi\frac{\sin 2\varphi}{\sqrt{\cos 2\varphi}}\right)d\varphi \overset{\text{adic. teor.}}{=} -a\frac{\sin 3\varphi}{\sqrt{\cos 2\varphi}}\,d\varphi$$

$$y' = \left(a\cos\varphi\sqrt{\cos 2\varphi} + a\sin\varphi\cdot\frac{1}{2}(\cos 2\varphi)^{-\frac{1}{2}}(-\sin 2\varphi)\cdot 2\right)d\varphi = \left(a\cos\varphi\sqrt{\cos 2\varphi} - a\sin\varphi\frac{\sin 2\varphi}{\sqrt{\cos 2\varphi}}\right)d\varphi \overset{\text{adiciona teor. ...}}{=} a\frac{\cos 3\varphi}{\sqrt{\cos 2\varphi}}\,d\varphi$$

$$x'^2+y'^2 = a^2\frac{\sin^2 3\varphi}{\cos 2\varphi}d\varphi^2 + a^2\frac{\cos^2 3\varphi}{\cos 2\varphi}d\varphi^2 = a^2\frac{1}{\cos 2\varphi}d\varphi^2 \Bigg| =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sqrt{\cos 2\varphi}\;a(\cos\varphi+\sin\varphi)\cdot a\frac{1}{\sqrt{\cos 2\varphi}}\,d\varphi = a^2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos\varphi+\sin\varphi)\,d\varphi =$$

$$= a^2\Big(\underbrace{\sin\varphi\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}}_{\frac{\sqrt{2}}{2}-\left(-\frac{\sqrt{2}}{2}\right)} - \underbrace{\cos\varphi\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}}_{\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}}\Big) = a^2\sqrt{2}$$ traženo rješenje.

Zadaci za vježbu

U zadacima 3770—3775 izračunati date krivolinijske integrale.

3770. $\int\limits_L \frac{ds}{x-y}$, pri čemu je L odsečak na pravoj $y=\frac{1}{2}x-2$, koji leži između tačaka $A(0, -2)$ i $B(4, 0)$.

3771. $\int\limits_L xy\,ds$, pri čemu je L kontura pravougaonika čija su temena $A(0, 0)$, $B(4, 0)$, $C(4, 2)$ i $D(0, 2)$.

3772. $\int\limits_L y\,ds$, pri čemu je L luk parabole $y^2=2px$, koji leži unutar parabole $x^2=2py$.

3773. $\int\limits_L (x^2+y^2)^n\,ds$, pri čemu je L krug $x=a\cos t$, $y=a\sin t$.

3774. $\int\limits_L xy\,ds$, pri čemu je L četvrtina elipse koja leži u prvom kvadrantu.

3775. $\int\limits_L \sqrt{2y}\,ds$, pri čemu je L prvi svod cikloide $x=a(t-\sin t)$,

$$y=a(1-\cos t).$$

3776. Napisati obrazac za izračunavanje integrala $\int\limits_L F(x, y)\,ds$ u polarnim koordinatama, ako je kriva L zadata jednačinom $\rho=\rho(\varphi)$ $(\varphi_1\leqslant\varphi\leqslant\varphi_2)$.

3777*. Izračunati $\int\limits_L (x-y)\,ds$, po kružnoj liniji $x^2+y^2=ax$.

3778. Izračunati $\int\limits_L \sqrt{x^2-y^2}\,ds$ po krivoj $(x^2+y^2)^2=a^2(x^2-y^2)$ $(x\geqslant 0)$ (polovina lemniskate).

3779. Izračunati $\int\limits_L \operatorname{arctg}\frac{y}{x}\,ds$ po delu Arhimedove spirale $\rho=2\varphi$ koji leži unutar kruga poluprečnika R, čiji je centar u koordinatnom početku.

3780. Izračunati $\int\limits_L \frac{z^2\,ds}{x^2+y^2}$ po prvom zavoju zavojnice $x=a\cos t$, $y=a\sin t$, $z=at$.

3781. Izračunati $\int\limits_L xyz\,ds$ po delu kružne linije $x^2+y^2+z^2=R^2$, $x^2+y^2=\frac{R^2}{4}$, koji leži u prvom oktantu.

3782. Izračunati $\int\limits_L (2z-\sqrt{x^2+y^2})\,ds$ po prvom zavoju konusne zavojnice $x=t\cos t$, $y=t\sin t$, $z=t$.

3783. Izračunati $\int\limits_L (x+y)\,ds$ po delu kružne linije $x^2+y^2+z^2=R^2$, $y=x$, koji leži u prvom oktantu.

Rješenja

3770. $\sqrt{5}\ln 2$. **3771.** 24.

3772. $\frac{p^2}{3}(5\sqrt{5}-1)$. **3773.** $2\pi a^{2n+1}$.

3774. $\frac{ab(a^2+ab+b^2)}{3(a+b)}$. **3775.** $4\pi a\sqrt{a}$.

3776. $\int\limits_{\varphi_1}^{\varphi_2} F(\rho\cos\varphi,\ \rho\sin\varphi)\sqrt{\rho^2+\rho'^2}\,d\varphi$.

3777*. $\frac{\pi a^2}{2}$. Preći na polarne koordinate.

3778. $\frac{2a^3\sqrt{2}}{3}$. **3779.** $\frac{1}{12}\left[(R^2+4)^{\frac{3}{2}}-8\right]$.

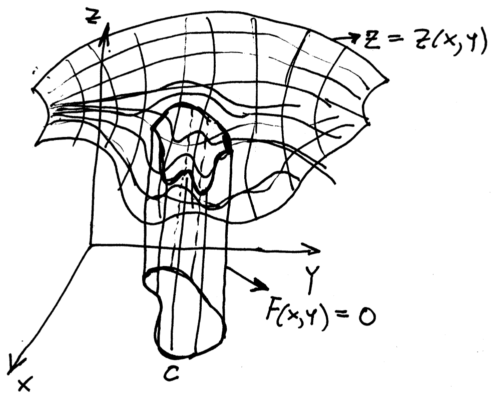
3780. $\frac{8a\pi^2\sqrt{2}}{3}$. **3781.** $\frac{R^4\sqrt{3}}{32}$

3782. $\frac{2\sqrt{2}}{3}\left[(1+2\pi^2)^{\frac{3}{2}}-1\right]$. **3783.** $R^2\sqrt{2}$.

Računanje površine cilindrične površi

Ako je S dio cilindrične površine $F(x,y)=0$ između xOy ravni i neke površine $z=z(x,y)$ tada se površina $P(S)$ površi S računa po formuli

$$P(S)=\int_C z(x,y)\,dS \quad \text{gdje je} \quad C:\begin{cases} F(x,y)=0 \\ z=0 \end{cases}.$$



$P(S)$ - površina dijela cilindrične površi

Izračunati površinu eliptičkog valjka $9x^2+5y^2=45$ koji se nalazi između površi $z=0$ i $z=y$.

Rj.

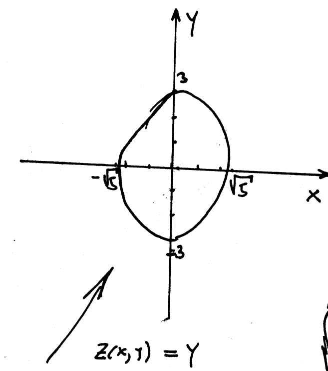
$$P(S)=\int_C z(x,y)\,dS \quad \text{gdje je} \quad C:\begin{cases} F(x,y)=0 \\ z=0 \end{cases}.$$

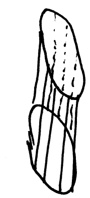
Skicirajmo valjak $9x^2+5y^2=45$. $/:45$ u xOy ravni on izgleda

$\frac{x^2}{5}+\frac{y^2}{9}=1$

elipsa $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

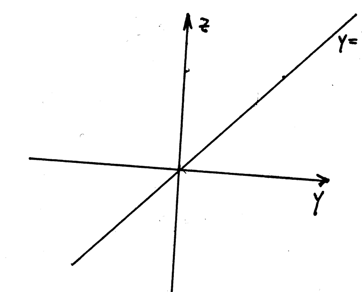
$\sqrt{5}\approx 2{,}23$





$z=0$ je xOy ravan

$z=y$ u yOz ravni



$z(x,y)=y$

$C:\begin{cases} 9x^2+5y^2=45 \\ z=0 \end{cases}$ C je elipsa

$z=y$ je ravan koja sadrži x-osu a u yOz ravni sadrži $y=z$ pravu

Svedimo elipsu $\frac{x^2}{5}+\frac{y^2}{9}=1$ na parametarski oblik $x=a\cos t$, $y=b\sin t$

U našem slučaju $x=\sqrt{5}\cos t$, $y=3\sin t$, $0\le t\le 2\pi$

površina figure koja se nalazi u I i II oktantu

$C:\begin{cases} x=\varphi(t) \\ y=\psi(t) \\ t_1\le t\le t_2 \end{cases}$ $\int_C f(x,y)\,ds=\int_{t_1}^{t_2} f(\varphi(t),\psi(t))\underbrace{\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt}_{ds}$

$dS=\sqrt{5\sin^2 t+9\cos^2 t}\,dt$ Kako se ravni $z=0$ i $z=y$ sijeku u x-osi to će parametar t uzimati vrijednosti od 0 do π

$$P(S)=\int_C y\,dS=\int_0^\pi 3\sin t\sqrt{5\sin^2 t+9\cos^2 t}\,dt=3\int_0^\pi \sin t\sqrt{5(1-\cos^2 t)+9\cos^2 t}\,dt=$$

$$=3\int_0^\pi \sin t\sqrt{5+4\cos^2 t}\,dt=\left|\begin{matrix} 2\cos t=u & t=0 \Rightarrow u=2 \\ -2\sin t\,dt=du & t=\pi \Rightarrow u=-2 \\ \sin t\,dt=-\frac{1}{2}du & \end{matrix}\right|=3\int_2^{-2}\left(-\frac{1}{2}\right)\underbrace{\sqrt{5+u^2}}_{\text{parna f-ja}}\,du=$$

$$=3\cdot\frac{1}{2}\cdot 2\int_0^2\sqrt{5+u^2}\,du \overset{\text{umjesto } u \text{ biramo } x}{=} 3\int_0^2\frac{5+x^2}{\sqrt{5+x^2}}\,dx=3\int_0^2\frac{5}{\sqrt{5+x^2}}\,dx+3\int_0^2\frac{x^2}{\sqrt{5+x^2}}=\left|\begin{matrix} u=x & dv=\frac{x}{\sqrt{5+x^2}} \\ \text{ZAVRŠITI SAMI} & \ldots \end{matrix}\right|=\frac{15\ln 5}{4}+9\ldots$$

(#) Izračunati površinu dijela valjka $x^2+y^2=1$ koji se nalazi između površi $z=0$ i $z=\sqrt{x^2+y^2}+\sqrt{1-x^2}+\sqrt{1-y^2}$

R: $P(S)=\int_C z(x,y)\,dS$ gdje je $c:\begin{cases}F(x,y)=0\\ z=0\end{cases}$

U ovom slučaju je $z(x,y)=\sqrt{x^2+y^2}+\sqrt{1-x^2}+\sqrt{1-y^2}$

$c:\begin{cases}x^2+y^2=1\\ z=0\end{cases}$ tj. $c: x^2+y^2=1$

Parametrizirajmo kružnicu: $x=r\cos\varphi$, $y=r\sin\varphi$, $0\le\varphi\le 2\pi$. U našem slučaju: $x=\cos\varphi$, $y=\sin\varphi$, $0\le\varphi\le 2\pi$

$c:\begin{cases}x=\mu(t)\\ y=\eta(t)\\ t_1\le t\le t_2\end{cases}$ $\int_C f(x,y)\,dS=\int_{t_1}^{t_2} f(\mu(t),\eta(t))\sqrt{(\mu'(t))^2+(\eta'(t))^2}\,dt$

$(\cos\varphi)'=-\sin\varphi$
$(\sin\varphi)'=\cos\varphi$

$dS=\sqrt{\sin^2\varphi+\cos^2\varphi}\,d\varphi=d\varphi$

$\sqrt{x^2+y^2}=1$; $\sqrt{1-x^2}=\cos\varphi$; $\sqrt{1-y^2}=\sin\varphi$

Definiciono područje f-je $z=\sqrt{x^2+y^2}+\sqrt{1-x^2}+\sqrt{1-y^2}$ je $\{(x,y)\,|\,-1\le x\le 1\ ;\ -1\le y\le 1\}$

zbog simetričnosti četiri dijela

$P(S)=\int_C\left(\sqrt{x^2+y^2}+\sqrt{1-x^2}+\sqrt{1-y^2}\right)dS=4\int_0^{\frac{\pi}{2}}(1+\sin\varphi+\cos\varphi)\,d\varphi=$

$=4\left[\varphi\Big|_0^{\frac{\pi}{2}}-\cos\varphi\Big|_0^{\frac{\pi}{2}}+\sin\varphi\Big|_0^{\frac{\pi}{2}}\right]=4\left(\frac{\pi}{2}+1+1\right)=2\pi+8$

(#) Izračunati površinu cilindra $x^2+y^2=R^2$ između ravni $z=0$ i površi $z=R+\frac{x^2}{R}$.

Zadaci za vježbu

U zadacima 3792 — 3797 izračunati površine datih cilindričnih omotača, koji leže između ravni *Oxy* i navedenih površina.

3792. $x^2+y^2=R^2, \quad z=R+\frac{x^2}{R}.$

3793. $y^2=2px, \quad z=\sqrt{2px-4x^2}.$

3794. $y^2=\frac{4}{9}(x-1)^3, \quad z=2-\sqrt{x}.$

3795. $x^2+y^2=R^2, \quad 2Rz=xy.$

3796. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad z=kx$ i $z=0$ $(z\geqslant 0)$ („cilindrična potkovica")

3797. $y=\sqrt{2px}, \quad z=y$ i $x=\frac{8}{9}p.$

3798. Izračunati površinu onog dela kružnog cilindra koji iz njega iseca drugi isti takav cilindar, ako im se ose seku pod pravim uglom a poluprečnici su im R (uporedi sa rešenjem zadatka 3642).

3799. Naći površinu onog dela cilindra $x^2+y^2=Rx$, koji leži unutar sfere $x^2+y^2+z^2=R^2$.

Rješenja

3792. $3\pi R^2$. **3793.** $\frac{\pi p^2}{4}$. **3794.** $\frac{11}{3}$. **3795.** R^2.

3796. $ka\left(a+\frac{b^2}{2c}\ln\frac{a+c}{a-c}\right)$, gde je $c=\sqrt{a^2-b^2}$. Za $a=b$ $S=2ka^2$.

3797. $\frac{98}{81}p^2$. **3798.** $8R^2$. **3799.** $4R^2$.

Krivolinijski integral druge vrste (po koordinatama)

Ako je c data kriva u ravni opisana jednačinom $y=\eta(x)$ gdje je $a \le x \le b$ tada

$$\int_c P(x,y)\,dx+Q(x,y)\,dy = \int_a^b \left[P(x,\eta(x)) + Q(x,\eta(x))\cdot \eta'(x)\right] dx$$

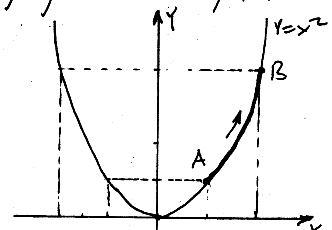
Ako je c data kriva opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$ gdje je $t_1 \le t \le t_2$ tada

$$\int_c P(x,y)\,dx+Q(x,y)\,dy = \int_{t_1}^{t_2} \left[P(\mu(t),\eta(t))\mu'(t)+Q(\mu(t),\eta(t))\eta'(t)\right] dt$$

Analogne formule vrijede za krivolinijski integral druge vrste uzete po prostornoj krivoj. Krivolinijski integral druge vrste OVISI O SMJERU PUTA INTEGRACIJE (bitna je orjentacija i u kom smjeru ide luk).

\# Izračunati krivolinijski integral $\int_c (x^2-2xy)\,dx+(2xy+y^2)\,dy$ gdje je c luk parabole $y=x^2$ od tačke $A(1,1)$ do $B(2,4)$.

Rj.



$y=x^2$

$\frac{dy}{dx}=2x$

$1 \le x \le 2$

Ako je data kriva $y=\eta(x)$, $a \le x \le b$

$$\int_c P(x,y)\,dx+Q(x,y)\,dy = \int_a^b \left[P(x,\eta(x))+Q(x,\eta(x))\cdot\eta'(x)\right]dx$$

$$\int_c (x^2-2xy)\,dx+(2xy+y^2)\,dy = \int_1^2 \left(x^2-2x^3+(2x^3+x^4)\,2x\right)dx = \int_1^2 (2x^5+4x^4-2x^3+x^2)\,dx$$

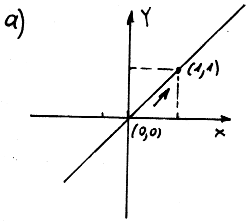
$$= 2\cdot\frac{1}{6}x^6\Big|_1^2 + 4\cdot\frac{1}{5}x^5\Big|_1^2 - 2\cdot\frac{1}{4}x^4\Big|_1^2 + \frac{1}{3}x^3\Big|_1^2 = \frac{1}{3}\cdot 63 + \frac{4}{5}\cdot 31 - \frac{1}{2}\cdot 15 + \frac{1}{3}\cdot 7 = 40+\frac{19}{30}$$

\# Izračunati krivolinijski integral $\int_{(0,0)}^{(1,1)} 2xy\,dx + x^2\,dy$ ako prelazimo po liniji

a) $y=x$ b) $y=x^2$ c) $y=x^3$ d) $y^2=x$.

Rj.

a)



Ako je data kriva $y=\eta(x)$, $\alpha \le x \le \beta$

$$\int_c P(x,y)\,dx+Q(x,y)\,dy = \int_\alpha^\beta \left[P(x,\eta(x))+Q(x,\eta(x))\cdot\eta'(x)\right]dx$$

$y=x$

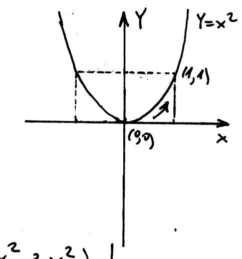
$y'=1$

$$\int_{(0,0)}^{(1,1)} 2xy\,dx+x^2\,dy = \int_0^1 (2x^2+x^2\cdot 1)\,dx = 3\int_0^1 x^2\,dx = 3\cdot\frac{1}{3}x^3\Big|_0^1 = 3\cdot\frac{1}{3} = 1$$

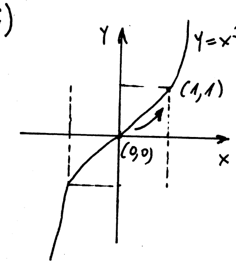
b) $y=x^2$

$y'=2x$

$$\int_{(0,0)}^{(1,1)} 2xy\,dx+x^2\,dy = \int_0^1 (2x\cdot x^2+x^2\cdot 2x)\,dx = \int_0^1 4x^3\,dx = 4\cdot\frac{1}{4}x^4\Big|_0^1 = 4\cdot\frac{1}{4} = 1$$



c)



$y=x^3$

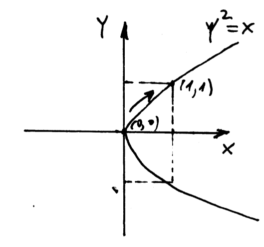
$y'=3x^2$

$$\int_{(0,0)}^{(1,1)} 2xy\,dx+x^2\,dy = \int_0^1 (2x\cdot x^3+x^2\cdot 3x^2)\,dx = \int_0^1 5x^4\,dx = 5\cdot\frac{1}{5}x^5\Big|_0^1 = 5\cdot\frac{1}{5} = 1$$

d) $x=y^2$

$x'_y=2y$

$$\int_{(0,0)}^{(1,1)} 2xy\,dx+x^2\,dy = \int_0^1 \left(2\cdot y^2\cdot y\cdot 2y+(y^2)^2\right)dy = \int_0^1 (4y^4+y^4)\,dy = \int_0^1 5y^4\,dy = 5\cdot\frac{1}{5}y^5\Big|_0^1 = 1$$



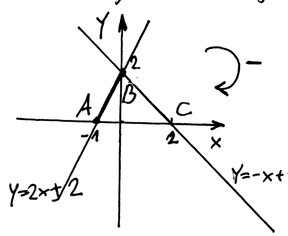
Izračunati krivoliniske integrale

a) $\oint_{-l} 2x\,dx-(x+2y)\,dy$

b) $\oint_{+l} y\cos x\,dx+\sin x\,dy$

gdje je l kontura trougla čiji su vrhovi $A(-1;0)$, $B(0;2)$ i $C(2;0)$.

Rj. a) Nacrtajmo trougao ΔABC.



Provucimo pravu kroz tačke $B(0;2)$ i $C(2;0)$

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$$

$$\frac{x}{2}=\frac{y-2}{-2}\quad /\cdot 2$$

$$x=-y+2$$

$$y=-x+2$$

Provucimo pravu kroz $A(-1;0)$ i $B(0;2)$

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}\qquad \frac{x+1}{1}=\frac{y}{2}$$

$$y=2x+2$$

$$\oint_{-l} 2x\,dx-(x+2y)\,dy=\int_{B(0;2)}^{C(2;0)} 2x\,dx-(x+2y)\,dy+\int_{C(2;0)}^{A(-1;0)} 2x\,dx-(x+2y)\,dy+\int_{A(-1;0)}^{B(0;2)} 2x\,dx-(x+2y)\,dy$$

$$\int_{(0;2)}^{(2;0)} 2x\,dx-(x+2y)\,dy=\begin{vmatrix} y=-x+2 \\ dy=-dx\end{vmatrix}=\int_{(0;2)}^{(2;0)}\left[2x-(x+2(-x+2))(-1)\right]dx=$$

$$=\int_{(0;2)}^{(2;0)}\left[2x+x-2x+4\right]dx=\int_{(0;2)}^{(2;0)}(x+4)\,dx=\left(\frac{1}{2}x^2+4x\right)\Big|_0^2=2+8=10$$

$$\int_{C(2;0)}^{A(-1;0)} 2x\,dx-(x+2y)\,dy=\begin{vmatrix} y=0 \\ dy=0\end{vmatrix}=\int_2^{-1}2x\,dx=2\cdot\frac{1}{2}x^2\Big|_2^{-1}=(1-4)=-3$$

$$\int_{A(-1;0)}^{B(0;2)} 2x\,dx-(x+2y)\,dy=\begin{vmatrix} y=2x+2 \\ dy=2dx\end{vmatrix}=\int_{-1}^{0}\left[2x-(x+2(2x+2))2\right]dx=$$

$$=\int_{-1}^{0}(2x-2x-8x-8)\,dx=(-8)\int_{-1}^{0}(x+1)\,dx=(-8)\left[\underbrace{\frac{1}{2}x^2\Big|_{-1}^{0}}_{0-1}+\underbrace{x\Big|_{-1}^{0}}_{0+1}\right]=$$

$$=(-8)\left(-\frac{1}{2}+1\right)=-4$$

Prema tome $\oint_{\Delta ABC} 2x\,dx-(x+2y)\,dy=10-3-4=3$

b) $$\oint_{+l} y\cos x\,dx+\sin x\,dy=\int_{AC} y\cos x\,dx+\sin x\,dy+\int_{CB} y\cos x\,dx+\sin x\,dy+\int_{BA} y\cos x\,dx+\sin x\,dy$$

$$\int_{A(-1;0)}^{C(2;0)} y\cos x\,dx+\sin x\,dy=\begin{vmatrix} y=0 \\ dy=0\end{vmatrix}=\int_{-1}^{2}0\,dx=0$$

$$\int_{C(2;0)}^{B(0;2)} y\cos x\,dx+\sin x\,dy=\begin{vmatrix} y=-x+2 \\ dy=-dx\end{vmatrix}=\int_2^0\left[(-x+2)\cos x-\sin x\right]dx$$

$$=\begin{vmatrix} u=-x+2 & dv=\cos x \\ du=-1 & v=\sin x\end{vmatrix}=(-x+2)\sin x\Big|_2^0+\int_2^0\sin x\,dx-\int_2^0\sin x\,dx=0$$

$$\int_{B(0;2)}^{A(-1;0)} y\cos x\,dx+\sin x\,dy=\begin{vmatrix} y=2x+2 \\ dy=2dx\end{vmatrix}=\int_0^{-1}\left[(2x+2)\cos x+2\sin x\right]dx=$$

$$=2\int_0^{-1}\left[(x+1)\cos x+\sin x\right]dx=\begin{vmatrix} u=x+1 & dv=\cos x \\ du=dx & v=\sin x\end{vmatrix}=2(x+1)\sin x\Big|_0^{-1}-2\int_0^{-1}\sin x\,dx$$

$$+2\int_0^{-1}\sin x\,dx=0$$

Prema tome

$$\oint_{+l} y\cos x\,dx+\sin x\,dy=0$$

Izračunati integral $I=\oint\limits_C y^2\,dx$

po krivoj koja nastaje kao presjek kugle i valjka

$$x^2+y^2+z^2=R^2, \qquad x^2+y^2=Rx.$$

Rj:

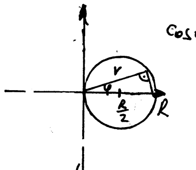
$$C:\begin{cases} x^2+y^2+z^2=R^2 \\ x^2+y^2=Rx \end{cases}$$

Posmatrajmo xOy ravan.
Prvo napišimo krug $x^2+y^2=Rx$ u parametarskom obliku.

$$x^2+y^2=Rx$$

$$x^2-2\cdot x\cdot\frac{R}{2}+\frac{R^2}{4}-\frac{R^2}{4}+y^2=0$$

$$\left(x-\frac{R}{2}\right)^2+y^2=\left(\frac{R}{2}\right)^2$$



$$\cos\varphi=\frac{r}{R}$$

$$\varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

Prisjetimo se polarnih koordinata

$$x=r\cos\varphi$$
$$y=r\sin\varphi$$

U našem slučaju za krug $x^2+y^2=Rx$ za r ćemo uzeti $r=R\cos\varphi$

Parametarski oblik kruga $x^2+y^2=Rx$ je

$$x=R\cos\varphi\cos\varphi=R\cos^2\varphi$$
$$y=R\cos\varphi\sin\varphi.$$

Uvrstimo ove vrijednosti u kuglu

$$x^2+y^2+z^2=R^2$$

$$\underbrace{R^2\cos^2\varphi\cos^2\varphi+R^2\cos^2\varphi\sin^2\varphi}+z^2=R^2$$

$$R^2\cos^2\varphi+z^2=R^2$$

$$z^2=R^2-R^2\cos^2\varphi$$

$$z^2=R^2(1-\cos^2\varphi)$$

$$z^2=R^2\sin^2\varphi$$

Parametarski oblik date krive je:

$$x=R\cos^2\varphi$$
$$y=R\cos\varphi\sin\varphi$$
$$z=R\sin\varphi$$
$$-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$$

$$I=\oint\limits_C y^2\,dx=\left|\begin{array}{l} x=R\cos^2\varphi \\ dx=R\,2\cos\varphi(-\sin\varphi)\,d\varphi \\ y=R\cos\varphi\sin\varphi \\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\end{array}\right|=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}R^2\cos^2\varphi\sin^2\varphi\cdot(-2)R\,\sin\varphi\cos\varphi\,d\varphi$$

$$=(-2)R^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^3\varphi\cos^3\varphi\,d\varphi=(-2)R^3\cdot\frac{1}{8}\int_{-\pi/2}^{\pi/2}(2\sin\varphi\cos\varphi)^3\,d\varphi=-\frac{1}{4}R^3\cdot\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin2\varphi)^3\,d(2\varphi)$$

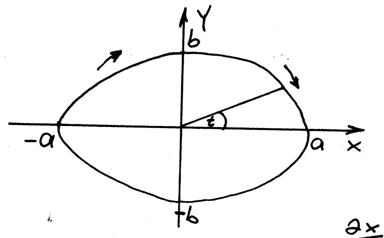
$$=-\frac{1}{8}R^3\int_{-\frac{\pi}{2}}^{\pi/2}\sin^2 2\varphi\cdot\sin2\varphi\,d(2\varphi)=-\frac{1}{8}R^3\int_{-\frac{\pi}{2}}^{\pi/2}(1-\cos^2 2\varphi)\sin2\varphi\,d(2\varphi)=$$

$$=+\frac{1}{8}R^3\int_{-\frac{\pi}{2}}^{\pi/2}(1-\cos^2 2\varphi)\cdot d(\cos2\varphi)=\frac{1}{8}R^3\Big(\underbrace{\cos2\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}_{-1+1}-\frac{1}{3}\underbrace{\cos^3 2\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}_{-1+1}\Big)=0$$

Izračunati krivoliniski integral $\int_c y^2dx + x^2dy$

gdje je c gornja polovina elipse $x=a\cos t$, $y=b\sin t$ $(a>0, b>0)$, koja se prelazi u smislu pomjeranja kazaljke na satu.

Rj.



Ako je kriva c zadana parametarski $x=\varphi(t)$, $y=\psi(t)$ gdje $\alpha\le t\le\beta$ imamo

$$\int_c P(x,y)\,dx + Q(x,y)\,dy = \int_\alpha^\beta \left[P(\varphi(t),\psi(t))\varphi'(t) + Q(\varphi(t),\psi(t))\psi'(t)\right]dt$$

$$\frac{\partial x}{\partial t} = -a\sin t \qquad \frac{\partial y}{\partial t} = b\cos t$$

$$\int_c y^2dx+x^2dy = \int_\pi^0 \left[b^2\sin^2 t\cdot(-a\sin t) + a^2\cos^2 t\cdot b\cos t\right]dt =$$

$$= -ab^2\int_\pi^0 \sin^3 t\,dt + a^2b\int_\pi^0 \cos^3 t\,dt \overset{(*)}{=} \frac{4}{3}ab^2$$

$$\int_\pi^0 \sin^3 t\,dt = \int_\pi^0 \sin t(1-\cos^2 t)\,dt = \left|\begin{matrix}\cos t = u & t=\pi \Rightarrow u=-1\\ -\sin t\,dt = du & t=0 \Rightarrow u=1\\ \sin t\,dt=-du & \end{matrix}\right| = -\int_{-1}^1 (1-u^2)\,du =$$

$$= -\left(u\Big|_{-1}^1 - \frac{1}{3}u^3\Big|_{-1}^1\right) = -\left(2-\frac{1}{3}\cdot 2\right) = -\left(\frac{6-2}{3}\right) = -\frac{4}{3} \quad \ldots(*)$$

$$\int_\pi^0 \cos^3 t\,dt = \int_\pi^0 \cos t(1-\sin^2 t)\,dt = \left|\begin{matrix}\sin t=u\\ \cos t\,dt=du\\ t=\pi \Rightarrow u=0\\ t=0 \Rightarrow u=0\end{matrix}\right| = \int_0^0 (1-u^2)\,du = 0$$

Date su tačke $A(3;-6;0)$ i $B(-2;4;5)$. Izračunati krivolinijski integral $I=\int_c xy^2dx + yz^2dy - zx^2dz$ gdje je c:

a) duž koja spaja tačke O i B (O koordinatni početak)

b) kriva od A do B: kruga zadan jednačinama $x^2+y^2+z^2=45$, $2x+y=0$.

Rj. $I=\int_c xy^2dx + yz^2dy + zx^2dz$

Ovo je krivoliniski integral druge vrste. Prisjetimo se:
Ako je c kriva u prostoru opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$, $z=\theta(t)$ gdje je $t_1\le t\le t_2$ tada

$$\int_c P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = \int_{t_1}^{t_2} P(\mu(t),\eta(t),\theta(t))\mu'(t)\,dt +$$
$$+\int_{t_1}^{t_2} Q(\mu(t),\eta(t),\theta(t))\eta'(t)\,dt + \int_{t_1}^{t_2} R(\mu(t),\eta(t),\theta(t))\theta'(t)\,dt$$

Da bi smo opisali duž $\overline{OB}$ u prostoru prvo postavimo pravu kroz ove dvije tačke.

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$ jednačina prave kroz dvije tačke $M_1(x_1,y_1,z_1)$ i $M_2(x_2,y_2,z_2)$

$O(0,0,0)$

$B(-2,4,5)$

$$\frac{x}{-2} = \frac{y}{4} = \frac{z}{5} \quad (=t)$$

$x=-2t$

$y=4t$

$z=5t$

Naše c je sada oblika $c: \begin{cases} x=-2t,\ y=4t,\ z=5t \\ 0<t<1\end{cases}$

$$I=\int_c xy^2dx + yz^2dy - zx^2dz = \int_0^1 \left((-2t)\,16t^2\cdot(-2) + 4t\cdot 25t^2\cdot 4 - 5t\cdot 4t^2\cdot 5\right)dt =$$

$$= \int_0^1 (64t^3 + 400t^3 - 100t^3)\,dt = 364\int_0^1 t^3\,dt = \frac{364}{4} = 91$$ traženo rješenje

b) Dat je krug u prostoru zadan jednačinama

$x^2+y^2+z^2=45$, $2x+y=0$

↑ krug ↑ ravan

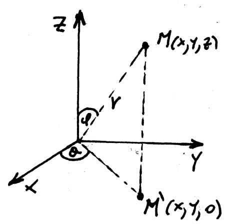
Da bi smo naš krug opisali u parametarskom obliku, veliku pomoć će odigrati sferne koordinate:

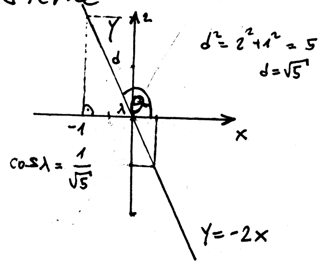
Sferne koordinate

$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$





Da bi smo krug u prostoru opisali parametarski potrebno je u sfernim koordinatama fiksirati r i θ. U našem slučaju ugao θ nije moguće svesti na lijep oblik.

Pristupimo parametarizaciji kruga na drugi način:

$2x+y=0 \Rightarrow y=-2x$

$x^2+y^2+z^2=45 \Rightarrow z^2=45-x^2-y^2$

$$\Rightarrow c: \begin{cases} x=t \\ y=-2t \\ z=\sqrt{45-t^2-4t^2}=\sqrt{45-5t^2} \\ 3\le t\le -2 \end{cases}$$

$dx = dt$, $dy=-2\,dt$, $dz = \frac{1}{2}(45-5t^2)^{-\frac{1}{2}}\cdot(-10t) = -\frac{5t}{\sqrt{45-5t^2}}\,dt$

$$I=\int_c xy^2dx + yz^2dy - zx^2dz = \int_3^{-2}\left(t\cdot 4t^2 + (-2t)(45-5t^2)\cdot(-2) - \sqrt{45-5t^2}\cdot t^2\,\frac{(-5t)}{\sqrt{45-5t^2}}\right)dt$$

$$= \int_3^{-2}(4t^3+180t-20t^3+5t^3)\,dt = \int_3^{-2}(-11t^3+180t)\,dt$$

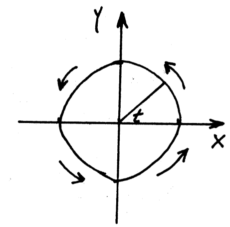
$$= -11\cdot\frac{1}{4}t^4\Big|_3^{-2} + 180\cdot\frac{1}{2}t^2\Big|_3^{-2} = -\frac{11}{4}\cdot(-65) + 90\cdot(-5) = \frac{715-1800}{4} = \frac{-1\,085}{4}$$

$$= -271\frac{1}{4}$$ traženo rješenje

Izračunati krivolinijski integral $\int_c \frac{(x+y)\,dx-(x-y)\,dy}{x^2+y^2}$

gdje je c krug $x^2+y^2=a^2$ koji se prelazi u smislu suprotnom pomjeranju kazaljke na satu.

R:



Krug $x^2+y^2=a^2$ napisan parametarski

$x = a\cos t$

$y = a\sin t$

$0\le t\le 2\pi$

$\frac{\partial x}{\partial t} = -a\sin t$

$\frac{\partial y}{\partial t} = a\cos t$

Ako je c kriva zadana parametarski $x=\mu(t)$, $y=\eta(t)$, $a\le t\le b$

$$\int_c P(x,y)\,dx + Q(x,y)\,dy = \int_a^b [P(\mu(t),\eta(t))\mu'(t) + Q(\mu(t),\eta(t))\eta'(t)]\,dt$$

$$\int_c \frac{(x+y)dx-(x-y)dy}{x^2+y^2} = \int_c \frac{x+y}{x^2+y^2}dx - \frac{x-y}{x^2+y^2}dy = \int_0^{2\pi}\left[\frac{a\cos t + a\sin t}{a^2}\cdot(-a\sin t) - \frac{a\cos t - a\sin t}{a^2}\cdot a\cos t\right]dt = \int_0^{2\pi}[(\cos t+\sin t)\cdot(-\sin t) - (\cos t - \sin t)\cdot\cos t]\,dt$$

$$= \int_0^{2\pi}(-\sin t\cos t - \sin^2 t - \cos^2 t + \sin t\cos t)\,dt = \int_0^{2\pi}(-1)\,dt = -2\pi$$

Izračunati krivolinijski integral $\int_C x^3dx + 3zy^2dy - x^2y\,dz$ gdje je C dio prave od tačke $A(3,2,1)$ do tačke $O(0,0,0)$.

Rj.

jednačina prave kroz dvije tačke $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

$p(O,A)$: $\frac{x}{3} = \frac{y}{2} = \frac{z}{1} \ (=t)$

$$\begin{cases} x = 3t \\ y = 2t \\ z = t \end{cases} \qquad \begin{aligned} dx &= 3\,dt \\ dy &= 2\,dt \\ dz &= dt \end{aligned}$$

Trebaju nam još granice za t

$A(3,2,1)$: $x=3t$, $y=2t$, $z=t$ $\Rightarrow t=1$

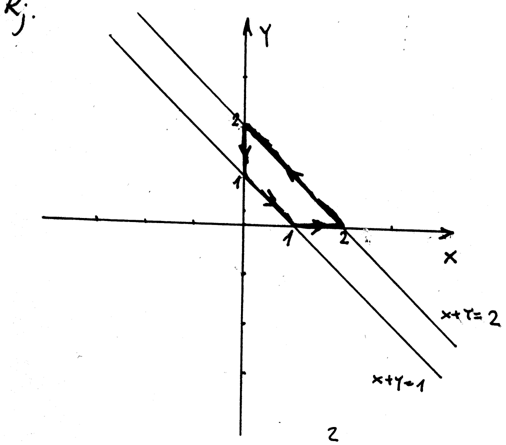
$O(0,0,0)$: $x=3t$, $y=2t$, $z=t$ $\Rightarrow t=0$

$$\int_C x^2dx + 3zy^2dy - x^2y\,dz = \int_1^0 \left[(3t)^3\cdot 3 + 3\cdot t\cdot(2t)^2\cdot 2 - (3t)^2\cdot 2t\cdot 1\right]dt$$

$$= \int_1^0 (81t^3 + 24t^3 - 18t^3)\,dt = -\int_0^1 87t^3\,dt = -87\cdot\frac{1}{4}t^4\Big|_0^1 = -\frac{87}{4}$$

Izračunati krivolinijski integral $I = \int_C (x^2+y^2)\,dx + x^2y\,dy$ gdje je c kontura trapeza koga obrazuju prave $x=0$, $y=0$, $x+y=1$, $x+y=2$.

Rj.



Ako je $c: y=\varphi(x)$, $a \le x \le b$

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_a^b \left[P(x,\varphi(x)) + Q(x,\varphi(x))\cdot\varphi'(x)\right]dx$$

U našem slučaju postoje 4 krive

C_1: $y=0$, $1 \le x \le 2$

C_1: $y=-x+2$, $2 \ge x \ge 0$

C_3: $x=0$, $2 \ge y \ge 1$

C_4: $y=-x+1$, $0 \le x \le 1$

$$I = I_1+I_2+I_3+I_4, \quad I_1 = \int_1^2 (x^2 + x^2\cdot 0)\,dx = \int_1^2 x^2dx = \frac{1}{3}x^3\Big|_1^2 = \frac{1}{3}(8-1) = \frac{7}{3}$$

$$I_2 = \int_2^0 \left(x^2+(-x+2)^2 + x^2(-x+2)\cdot(-1)\right)dx = \int_2^0 (x^2+x^2-4x+4+x^3-2x^2)\,dx =$$

$$= \int_2^0 (x^3-4x+4)\,dx = \frac{1}{4}x^4\Big|_2^0 - 4\cdot\frac{1}{2}x^2\Big|_2^0 + 4x\Big|_2^0 = -4+8-8 = -4$$

$$I_3 = \int_2^1 (y^2\cdot 0 + 0y)\,dy = 0$$

$$I_4 = \int_0^1 \left(x^2+(-x+1)^2 + x^2(-x+1)\cdot(-1)\right)dx = \int_0^1 (x^2+x^2-2x+1+x^3-x^2)\,dx =$$

$$= \int_0^1 (x^3+x^2-2x+1)\,dx = \frac{1}{4}x^4\Big|_0^1 + \frac{1}{3}x^3\Big|_0^1 - 2\cdot\frac{1}{2}x^2\Big|_0^1 + x\Big|_0^1 = \frac{1}{4}+\frac{1}{3}-1+1 = \frac{7}{12}$$

$$I = I_1+I_2+I_3+I_4 = \frac{7}{3} + (-4) + \frac{7}{12} = \frac{28-48+7}{12} = -\frac{13}{12}$$ vrijednost krivolinijskog integrala

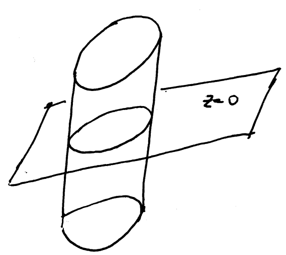
II način: Greenova formula ...

(#) Izračunati krivolinijski integral

$$I = \oint_C y\,dx + x^2\,dy$$

duž krive koja nastaje kao presjek ravni $z=0$ i cilindra $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{a} + \frac{y}{b}$ orjentisana u pozitivnom smjeru $(a \geq b > 0)$.

Rj. Za rješenje zadatka nije nam bitno gdje se cilindar nalazi u prostoru



$$z=0$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{a} + \frac{y}{b}$$

$$\frac{1}{a^2}\left(x^2 - ax\right) + \frac{1}{b^2}\left(y^2 - by\right) = 0$$

$$\frac{1}{a^2}\left(x^2 - 2\cdot x\cdot\frac{a}{2} + \frac{a^2}{4} - \frac{a^2}{4}\right) + \frac{1}{b^2}\left(y^2 - 2\cdot y\cdot\frac{b}{2} + \frac{b^2}{4} - \frac{b^2}{4}\right) = 0$$

$$\frac{1}{a^2}\left(x - \frac{a}{2}\right)^2 - \frac{1}{4} + \frac{1}{b^2}\left(y - \frac{b}{2}\right)^2 - \frac{1}{4} = 0$$

$$\frac{\left(x-\frac{a}{2}\right)^2}{a^2} + \frac{\left(y-\frac{b}{2}\right)^2}{b^2} = \frac{1}{2} \quad /\cdot 2$$

$$\frac{\left(x-\frac{a}{2}\right)^2}{\frac{a^2}{2}} + \frac{\left(y-\frac{b}{2}\right)^2}{\frac{b^2}{2}} = 1$$ ovo je elipsa

Elipsu ćemo parametrizirati pomoću pomjerenih polarnih koordinata

$$x = \frac{a}{2} + \frac{a}{\sqrt{2}}\cos\varphi \qquad dx = -\frac{a}{\sqrt{2}}\sin\varphi\, d\varphi$$

$$y = \frac{b}{2} + \frac{b}{\sqrt{2}}\sin\varphi \qquad dy = \frac{b}{\sqrt{2}}\cos\varphi\, d\varphi$$

$$0 \leq \varphi < 2\pi$$

Sad nije teško izračunati dati krivolinijski integral

$$I = \oint_C y\,dx + x^2\,dy = \int_0^{2\pi}\left[\left(\frac{b}{2} + \frac{b}{\sqrt{2}}\sin\varphi\right)\frac{-a}{\sqrt{2}}\sin\varphi + \left(\frac{a}{2} + \frac{a}{\sqrt{2}}\cos\varphi\right)^2\cdot\frac{b}{\sqrt{2}}\cos\varphi\right]d\varphi$$

$$= \frac{-a}{\sqrt{2}}\int_0^{2\pi}\left(\frac{b}{2} + \frac{b}{\sqrt{2}}\sin\varphi\right)\sin\varphi\, d\varphi + \frac{b}{\sqrt{2}}\int_0^{2\pi}\left(\frac{a^2}{4} + \frac{a^2}{\sqrt{2}}\cos\varphi + \frac{a^2}{2}\cos^2\varphi\right)\cos\varphi\, d\varphi$$

$$= \frac{-ab}{2\sqrt{2}}\int_0^{2\pi}\sin\varphi\, d\varphi - \frac{ab}{2}\int_0^{2\pi}\underbrace{\sin^2\varphi}_{\frac{1}{2}(1-\cos 2\varphi)}\, d\varphi + \frac{a^2 b}{4\sqrt{2}}\int_0^{2\pi}\cos\varphi\, d\varphi + \frac{a^2 b}{2}\int_0^{2\pi}\underbrace{\cos^2\varphi}_{\frac{1}{2}(1+\cos 2\varphi)}\, d\varphi + \frac{a^2 b}{2\sqrt{2}}\int_0^{2\pi}\cos^3\varphi\, d\varphi$$

$$= \frac{-ab}{2\sqrt{2}}\underbrace{\cos\varphi\Big|_0^{2\pi}}_{=1-1=0} - \frac{ab}{2}\cdot\frac{1}{2}\left(\varphi\Big|_0^{2\pi} - \underbrace{\frac{1}{2}\sin 2\varphi\Big|_0^{2\pi}}_{=0}\right) + 0 + \frac{a^2 b}{2}\cdot\frac{1}{2}\left(\varphi\Big|_0^{2\pi} + \underbrace{\frac{1}{2}\sin 2\varphi\Big|_0^{2\pi}}_{=0}\right)$$

$$+ \frac{a^2 b}{2\sqrt{2}}\int_0^{2\pi}\cos^2\varphi\cos\varphi\, d\varphi =$$

$$= -\frac{ab\pi}{2} + \frac{a^2 b\pi}{2} + \frac{a^2 b}{2\sqrt{2}}\underbrace{\int_0^{2\pi}\left(1 - \sin^2\varphi\right)d(\sin\varphi)}_{=0}$$

$$= \frac{\pi ab(-1+a)}{2} = \frac{ab\pi}{2}(a-1)$$

traženo rješenje

$1 = \sin^2\varphi + \cos^2\varphi$

$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi$

$1 - \cos 2\varphi = 2\sin^2\varphi$

$\sin^2\varphi = \frac{1}{2}(1 - \cos 2\varphi)$

$1 + \cos 2\varphi = 2\cos^2\varphi$

II način: Greenova formula ...

Izračunati krivoliniski integral

$$I=\oint_C z\,dz.$$

duž krive koja nastaje kao presjek cilindra $\frac{(x-\frac{a}{2})^2}{\frac{a^2}{2}}+\frac{(y-\frac{b}{2})^2}{\frac{b^2}{2}}=1$ i paraboloida $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ orjentisana u pozitivnom smjeru $(a\geq b>0)$.

Rj. Prisjetimo se

Ako je kriva $C:\begin{cases} x=\mu(t)\\ y=\eta(t)\\ z=\theta(t)\\ t_1\leq t\leq t_2\end{cases}$ data u parametarskom obliku, tada

$$\int_C P(x,y,z)\,dx+Q(x,y,z)\,dy+R(x,y,z)\,dz=\int_{t_1}^{t_2}\Big(P(\mu(t),\eta(t),\theta(t))\,\mu'(t)+Q(\mu(t),\eta(t),\theta(t))\,\eta'(t)+R(\mu(t),\eta(t),\theta(t))\,\theta'(t)\Big)\,dt$$

Da bi izračunali dati integral trebamo parametrizirati datu krivu. Uvrstimo smjene za x i y t.d. $\frac{(x-\frac{a}{2})^2}{\frac{a^2}{2}}+\frac{(y-\frac{b}{2})^2}{\frac{b^2}{2}}=1$. $(*)$

Za x i y mogu nam pomoći poopštene polarne koordinate (gdje je r fiksirano)

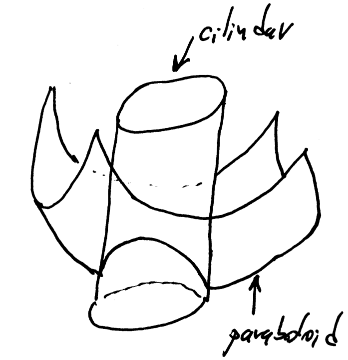
$$\left.\begin{aligned} x&=\frac{a}{2}+\frac{a}{\sqrt2}\cos\varphi &&\Rightarrow \left(x-\frac{a}{2}\right)^2=\frac{a^2}{2}\cos^2\varphi\\ y&=\frac{b}{2}+\frac{b}{\sqrt2}\sin\varphi &&\Rightarrow \left(y-\frac{b}{2}\right)^2=\frac{b^2}{2}\sin^2\varphi\end{aligned}\right\}\Rightarrow \text{vrijedi } (*) \text{ za } \varphi\in[0,2\pi)$$

Sada je

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{\left(\frac{a}{2}+\frac{a}{\sqrt2}\cos\varphi\right)^2}{a^2}+\frac{\left(\frac{b}{2}+\frac{b}{\sqrt2}\sin\varphi\right)^2}{b^2}=\left(\frac12+\frac{1}{\sqrt2}\cos\varphi\right)^2+\left(\frac12+\frac{1}{\sqrt2}\sin\varphi\right)^2=$$

$$=\frac14+\frac{1}{\sqrt2}\cos\varphi+\frac12\cos^2\varphi+\frac14+\frac{1}{\sqrt2}\sin\varphi+\frac12\sin^2\varphi=$$

$$=1+\frac{1}{\sqrt2}\cos\varphi+\frac{1}{\sqrt2}\sin\varphi.$$



Prema tome imamo

$$z=1+\frac{1}{\sqrt2}\cos\varphi+\frac{1}{\sqrt2}\sin\varphi$$

$$dz=-\frac{1}{\sqrt2}\sin\varphi+\frac{1}{\sqrt2}\cos\varphi$$

$$\oint_C z\,dz=\int_0^{2\pi}\left(1+\frac{1}{\sqrt2}\cos\varphi+\frac{1}{\sqrt2}\sin\varphi\right)\left(-\frac{1}{\sqrt2}\sin\varphi+\frac{1}{\sqrt2}\cos\varphi\right)d\varphi=$$

$$=\int_0^{2\pi}\Big(-\frac{1}{\sqrt2}\sin\varphi+\frac{1}{\sqrt2}\cos\varphi-\frac12\sin\varphi\cos\varphi+\underbrace{\frac12\cos^2\varphi-\frac12\sin^2\varphi}_{=\frac12(\cos^2\varphi-\sin^2\varphi)}+\frac12\sin\varphi\cos\varphi\Big)\,d\varphi$$

$$=-\frac{1}{\sqrt2}\int_0^{2\pi}\sin\varphi\,d\varphi+\frac{1}{\sqrt2}\int_0^{2\pi}\cos\varphi\,d\varphi+\frac12\int_0^{2\pi}\cos2\varphi\,d\varphi=$$

$$=-\frac{1}{\sqrt2}(-\cos\varphi)\Big|_0^{2\pi}+\frac{1}{\sqrt2}\sin\varphi\Big|_0^{2\pi}+\frac12\cdot\frac12\sin2\varphi\Big|_0^{2\pi}=$$

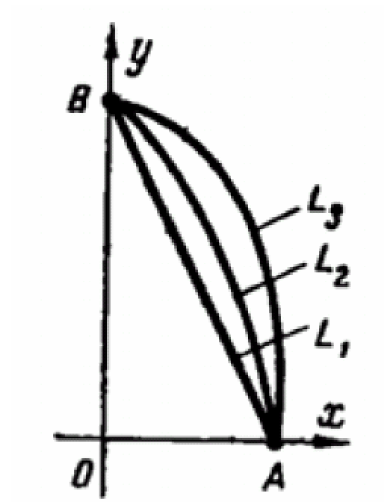
$$=\frac{1}{\sqrt2}(1-1)+0+0=0$$

1. Izračunaj krivolinijski integral $I=\int\limits_L (xy-1)dx + x^2 y\,dy$ od tačke A(1,0) do tačke B(0,2).

a) po pravoj 2x+y=2
b) duž parabole $4x+y^2=4$
c) duž elipse x=cost ; y= 2sint

Rješenja:

a) Skicirajmo datu pravu (uputa vidi sliku desno).



2x+y=2
y = 2 - 2x
dy = -2dx

$$I=\int\limits_{L_1}(xy-1)dx+x^2y\,dy=$$

$$=\int\limits_1^0\left[x(2-2x)-1\right]dx+x^2(2-2x)(-2dx)=$$

$$=\int\limits_1^0(2x-2x^2-1)dx+(-4x^2+4x^3)dx=$$

$$=\int\limits_1^0(4x^3-6x^2+2x-1)dx=4\cdot\frac{x^4}{4}\Big|_1^0-6\cdot\frac{x^3}{3}\Big|_1^0+2\cdot\frac{x^2}{2}\Big|_1^0-x\Big|_1^0=-1+2-1+1=1$$

b) Skicirajmo parabolu (uputa: vidi sliku iznad).

$$4x+y^2=4\ \Rightarrow\ x=1-\frac{y^2}{4}\ \Rightarrow\ dx=-\frac{y}{2}dy$$

$$I=\int\limits_{L_2}(xy-1)dx+x^2y\,dy=\int\limits_0^2\left[\left(1-\frac{y^2}{4}\right)y-1\right]\left(-\frac{y}{2}dy\right)+\left(1-\frac{y^2}{4}\right)^2 y\,dy=$$

$$=\int\limits_0^2\left(y-\frac{y^3}{4}-1\right)\left(-\frac{y}{2}dy\right)+\left(1-\frac{y^2}{2}+\frac{y^4}{16}\right)y\,dy=$$

$$=\int\limits_0^2\left(-\frac{y^2}{2}+\frac{y^4}{8}+\frac{y}{2}\right)dy+\left(y-\frac{y^3}{2}+\frac{y^5}{16}\right)dy=$$

$$=\int\limits_0^2\left(\frac{y^5}{16}+\frac{y^4}{8}-\frac{y^3}{2}-\frac{y^2}{2}+\frac{3y}{2}\right)dy=\frac{y^6}{96}\Big|_0^2+\frac{y^5}{40}\Big|_0^2-\frac{y^4}{8}\Big|_0^2-\frac{y^3}{6}\Big|_0^2+\frac{3y^2}{4}\Big|_0^2=$$

$$=\frac{64}{96}+\frac{32}{40}-\frac{16}{8}-\frac{8}{6}+\frac{12}{4}=\frac{2}{3}+\frac{4}{5}-2-\frac{4}{3}+3=\frac{10+12-30-20+45}{15}=\frac{17}{15}.$$

c) Skicirajmo elipsu (uputa: vidi sliku sa prethodne stranice).

x = cost y = 2sint

dy = 2costdt

$$L_3:\begin{cases}x=\cos t\\ y=\sin t\\ 0\le t\le\frac{\pi}{2}\end{cases}$$

$$I=\int\limits_{L_3}(xy-1)dx+x^2y\,dy=\int\limits_0^{\frac{\pi}{2}}(\cos t\cdot 2\sin t-1)\cdot(-\sin t\,dt)+\cos^2 t\cdot 2\sin t\cdot 2\cos t\,dt=$$

$$=\int\limits_0^{\frac{\pi}{2}}(-2\sin^2 t\cos t+\sin t)dt+4\cos^3 t\sin t\,dt=\int\limits_0^{\frac{\pi}{2}}(4\cos^3 t\sin t+\sin t-2\sin^2 t\cos t)dt=$$

$$=4\int\limits_0^{\frac{\pi}{2}}\cos^3 t\sin t\,dt+\int\limits_0^{\frac{\pi}{2}}\sin t\,dt-2\int\limits_0^{\frac{\pi}{2}}\sin^2 t\cos t\,dt\overset{*}{=}$$

$$\int\cos^3 t\sin t\,dt=\left|\begin{matrix}\cos t=u\\ -\sin t\,dt=du\\ \sin t\,dt=-du\end{matrix}\right|=-\int u^3du=-\frac{u^4}{4}+c=-\frac{\cos^4 t}{4}+c$$

$$\int\sin t\cos t\,dt=\left|\begin{matrix}\sin t=u\\ \cos t\,dt=du\end{matrix}\right|=\int u^2du=\frac{u^3}{3}+c=\frac{\sin^3 t}{3}+c$$

$$\overset{*}{=}4\cdot\left(-\frac{\cos^4 t}{4}\right)\Bigg|_0^{\frac{\pi}{2}}-\cos t\Big|_0^{\frac{\pi}{2}}-2\left(\frac{\sin^3 t}{3}\right)\Bigg|_0^{\frac{\pi}{2}}=4\cdot\frac{1}{4}+1-2\cdot\frac{1}{3}=2-\frac{2}{3}=\frac{4}{3}$$

Zadaci za vježbu

U zadacima 3806 — 3821 izračunati date krivolinijske integrale.

3806. $\int_L x\,dy$ po konturi trougla koji obrazuju koordinatne ose i prava $\frac{x}{2}+\frac{y}{3}=1$, — u pozitivnom smeru obilaženja (tj. nasuprot kretanju satne kazaljke).

3807. $\int_L x\,dy$ po odsečku prave $\frac{x}{a}+\frac{y}{b}=1$, od tačke preseka sa apscisnom do tačke preseka sa ordinatnom osom.

3808. $\int_L (x^2-y^2)\,dx$ po delu parabole $y=x^2$ od koordinatnog početka do tačke (2, 4).

3809. $\int_L (x^2+y^2)\,dy$ po konturi četvorougla čija su temena (navedena po redu obilaženja): $A(0, 0)$, $B(2, 0)$, $C(4, 4)$ i $D(0, 4)$.

3810. $\int_{(0,0)}^{(\pi, 2\pi)} -x\cos y\,dx + y\sin x\,dy$ duž pravolinijskog odsečka koji spaja tačke (0, 0) i (π, 2π).

3811. $\int_{(0,0)}^{(1,1)} xy\,dx+(y-x)\,dy$ duž krive 1) $y=x$, 2) $y=x^2$, 3) $y^2=x$, 4) $y=x^3$.

3812. $\int_{(0,0)}^{(1,1)} 2xy\,dx+x^2\,dy$ duž krive 1) $y=x$, 2) $y=x^2$, 3) $y=x^2$, 4) $y^2=x$.

3813. $\int_L y\,dx+x\,dy$ po delu kruga $x=R\cos t$, $y=R\sin t$, od $t_1=0$ do $t_2=\frac{\pi}{2}$.

3814. $\int_L y\,dx-x\,dy$ po elipsi $x=a\cos t$, $y=b\sin t$, u pozitivnom smeru obilaženja.

3815. $\int_L \frac{y^2\,dx-x^2\,dy}{x^2+y^2}$, po polukrugu $x=a\cos t$, $y=a\sin t$ od $t_1=0$ do $t_2=\pi$.

3816. $\int_L (2a-y)\,dx-(a-y)\,dy$ duž prvog (računajući od koordinatnog početka) svoda cikloide $x=a(t-\sin t)$, $y=a(1-\cos t)$.

3817. $\int_L \frac{x^2\,dy-y^2\,dx}{x^{\frac{5}{3}}+y^{\frac{5}{3}}}$, pri čemu je L deo astroide $x=R\cos^3 t$, $y=R\sin^3 t$ od tačke $(R, 0)$ do tačke $(0, R)$.

3818. $\int_L x\,dx+y\,dy+(x+y-1)\,dz$ duž pravolinijskog odsečka od tačke (1, 1, 1) do tačke (2, 3, 4).

3819. $\int_L yz\,dx+z\sqrt{R^2-y^2}\,dy+xy\,dz$ po zavojnici $x=R\cos t$, $y=R\sin t$, $z=\frac{at}{2\pi}$, od njenog preseka sa ravni $z=0$ do preseka sa ravni $z=a$.

3820. $\int_{(1,1,1)}^{(4,4,4)} \frac{x\,dx+y\,dy+z\,dz}{\sqrt{x^2+y^2+z^2-x-y+2z}}$ duž prave linije.

3821. $\int_L y^2\,dx+z^2\,dy+x^2\,dz$ duž krive po kojoj se seku sfera $x^2+y^2+z^2=R^2$ i cilindar $x^2+y^2=Rx$ ($R>0$, $z\geqslant 0$), pri čemu je smer obilaženja po konturi, posmatran iz koordinatnog početka, suprotan kretanju satne kazaljke.

Rješenja

3806. 3. **3807.** $\frac{ab}{2}$.

3808. $-\frac{56}{15}$. **3809.** $37\frac{1}{3}$.

3810. 4π. **3811.** 1) $\frac{1}{3}$; 2) $\frac{1}{12}$; 3) $\frac{17}{30}$; 4) $-\frac{1}{20}$.

3812. U sva četiri slučaja vrednost integrala je 1.

3813. 0. **3814.** $-2\pi ab$.

3815. $\frac{4}{3}a$. **3816.** πa^2.

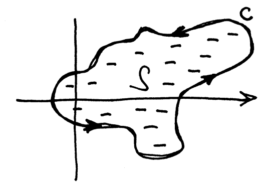
3817. $\frac{3}{16}\pi R\sqrt[3]{R}$

3818. 13. **3819.** $-\frac{a\pi R^4}{2}$

3820. $3\sqrt{3}$. **3821.** $-\frac{\pi R^3}{4}$.

Greenova formula za ravan

Ako je c po djelovima glatka granica područja S, a f-je $P(x,y)$ i $Q(x,y)$ neprekidne zajedno sa svojim parcijalnim izvodima prvog reda u zatvorenom području S+C, onda vrijedi Greenova formula



$$\int_C P\,dx+Q\,dy=\iint_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

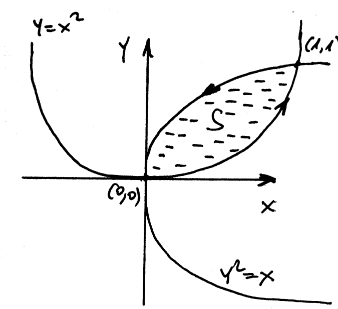
c - zatvorena kontura

S - oblast ograničena konturom

Izračunati integral $\int_C (2xy-x^2)\,dx+(x+y^2)\,dx$

gdje je c kontura površine ograničene sa $y=x^2$ i $y^2=x$.

Rj.



$P(x,y)=2xy-x^2$ $\qquad \frac{\partial P}{\partial y}=2x$

$Q(x,y)=x+y^2$ $\qquad \frac{\partial Q}{\partial x}=1$.

$$\int_C P\,dx-Q\,dy=\iint_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

formula Greena

$$\int_C (2xy-x^2)\,dx+(x+y^2)\,dx=\iint_S (1-2x)\,dx\,dy=\int_0^1\left[\int_{x^2}^{\sqrt{x}}(1-2x)\,dy\right]dx=$$

$$=\int_0^1\left(y\Big|_{x^2}^{\sqrt{x}}-2xy\Big|_{x^2}^{\sqrt{x}}\right)dx=\int_0^1\left(\sqrt{x}-x^2-2x(\sqrt{x}-x^2)\right)dx=$$

$$=\int_0^1\left(2x^3-x^2-2x^{\frac{3}{2}}+x^{\frac{1}{2}}\right)=2\cdot\frac{1}{4}x^4\Big|_0^1-\frac{1}{3}x^3\Big|_0^1-2\cdot\frac{2}{5}x^{\frac{5}{2}}\Big|_0^1+\frac{2}{3}x^{\frac{3}{2}}\Big|_0^1=\frac{1}{30}$$

(#) Izračunati krivoliniske integrale

a) $\oint_{-\ell} 2x\,dx-(x+2y)\,dy$ i b) $\oint_{+\ell} y\cos x\,dx+\sin x\,dy$

po krivoj ℓ, gdje je ℓ trougao čiji su vrhovi $A(-1;0)$, $B(0;2)$ i $C(2;0)$.

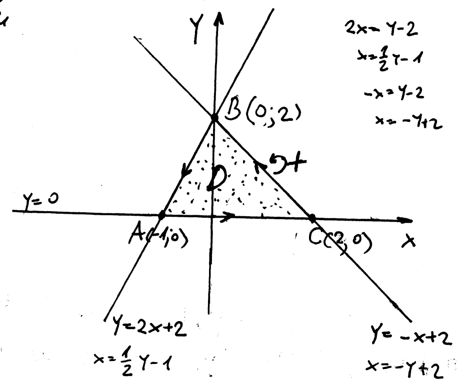
Rj.

$\int_c P(x,y)\,dx+Q(x,y)\,dy$ je krivoliniski integral druge vrste.

Ako je kriva c data u obliku $y=\eta(x)$, $c_1\le x\le c_2$ dati integral se računa po formuli

$$\int_{c_1}^{c_2}\Big(P(x,\eta(x))+Q(x,\eta(x))\,\eta'(x)\Big)\,dx$$

Skicirajmo tačke u xOy ravni



prava koja prolazi kroz tačke A i B je

$$\frac{x}{-1}+\frac{y}{2}=1 \quad /\cdot 2$$

$$-2x+y=1$$

$$y=2x+2 \Rightarrow y'=2$$

prava koja prolazi kroz tačke B i C

$$\frac{x}{2}+\frac{y}{2}=1 \quad /\cdot 2$$

$$x+y=2$$

$$y=-x+2 \Rightarrow y'=-1$$

a) $$\oint_{-\ell} 2x\,dx-(x+2y)\,dy=\int_{AB} 2x\,dx-(x+2y)\,dy+\int_{BC} 2x\,dx-(x+2y)\,dy+\int_{CA} 2x\,dx-(x+2y)\,dy$$

AB po pravoj $y=2x+2$; BC po pravoj $y=-x+2$; CA po pravoj $y=0$

$$\int_{AB} 2x\,dx-(x+2y)\,dy=\int_{-1}^{0}\big[2x-\underbrace{(x+2(2x+2))2}_{(x+4x+4)\cdot 2 \;=\; 10x+8}\big]\,dx=\int_{-1}^{0}(-8x-8)\,dx=-8\cdot\frac{1}{2}x^2\Big|_{-1}^{0}-8x\Big|_{-1}^{0}=(-4)(-1)-8=-4$$

$$\int_{BC} 2x\,dx-(x+2y)\,dy=\int_{0}^{2}\big[2x-\underbrace{(x+2(-x+2))(-1)}_{(x-2x+4)(-1) \;=\; x-4}\big]\,dx=\int_{0}^{2}(x+4)\,dx=\frac{1}{2}x^2\Big|_0^2+4x\Big|_0^2=2+8=10$$

BC po pravoj $y=-x+2$

$$\int_{CA} 2x\,dx-(x+2y)\,dy=\int_{2}^{-1}\big[2x-(x+2(0))0\big]\,dx=\int_{2}^{-1}2x\,dx=2\cdot\frac{1}{2}x^2\Big|_2^{-1}=1-4=-3$$

CA po pravoj $y=0$

$$\oint_{-\ell} 2x\,dx-(x+2y)\,dy=-4+10-3=3$$ traženo rješenje

b) Možemo upotrebiti Greenovu formulu

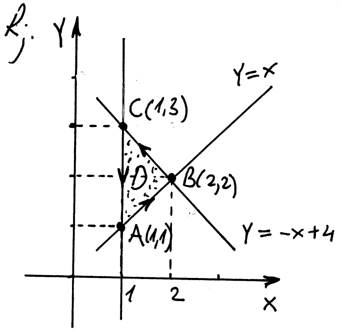
$$\int_c P(x,y)\,dx+Q(x,y)\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$ gdje je D oblast ograničena konturom c

$$\oint_{+\ell} y\cos x\,dx+\sin x\,dy=\left|\begin{array}{ll} Q(x,y)=\sin x & P(x,y)=y\cos x \\ \frac{\partial Q}{\partial x}=\cos x & \frac{\partial P}{\partial y}=\cos x \\ \text{D-vidi sliku (tačkasti dio sa slike)} & \end{array}\right|=$$

$$=\iint_D(\cos x-\cos x)\,dx\,dy=\iint_D 0\,dx\,dy=0$$ traženo rješenje

Izračunati $\int_c 2(x^2+y^2)dx+(x+y)^2dy$ gdje je c kontura trougla ΔABC pozitivno orjentisana $(A(1,1),\ B(2,2),\ C(1,3))$.

Rj.



$P(x,y) = 2(x^2+y^2) = 2x^2+2y^2$

$Q(x,y) = (x+y)^2 = x^2+2xy+y^2$

$$\int_c P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$ formula Grina

$y - y_1 = \frac{x_2-x_1}{y_2-y_1}(x-x_1)$

$y-2 = \frac{-1}{1}(x-2)$

$y-2 = -x+2 \Rightarrow y = -x+4$

$\frac{\partial P}{\partial y} = 4y \qquad \frac{\partial Q}{\partial x} = 2x+2y \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x+2y-4y = 2x-2y$

$D:\begin{cases} 1 \le x \le 2 \\ x \le y \le 4-x \end{cases}$

$$\int_c 2(x^2+y^2)dx + (x+y)^2dy = \iint_D (2x-2y)\,dx\,dy =$$

$$= \int_1^2 \left[\int_x^{4-x} (2x-2y)\,dy\right] dx = \int_1^2 \left(2xy\Big|_x^{4-x} - 2\cdot\frac{1}{2}y^2\Big|_x^{4-x}\right) dx =$$

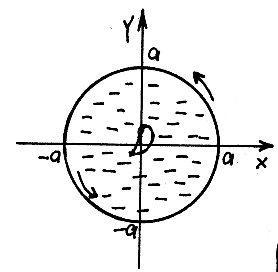
$4-x-x \qquad (4-x)^2-x^2$

$$= \int_1^2 \big(2x(4-2x) - (16-8x)\big)\,dx = \int_1^2 (8x-4x^2-16+8x)\,dx = \int_1^2 (-4x^2+16x-16)\,dx$$

$$= -4\cdot\frac{1}{3}x^3\Big|_1^2 + 16\cdot\frac{1}{2}x^2\Big|_1^2 - 16x\Big|_1^2 = -\frac{4}{3}\cdot 7 + 8\cdot 3 - 16 = 8 - \frac{28}{3} = -\frac{4}{3}$$

Izračunati $\int_c xy^2dy - x^2y\,dx$ gdje je c krug $x^2+y^2=a^2$. Integraciju izvesti u pozitivnom smjeru.

Rj.



$P(x,y) = -x^2y \qquad \frac{\partial P}{\partial y} = -x^2$

$Q(x,y) = xy^2 \qquad \frac{\partial Q}{\partial x} = y^2 \qquad D: x^2+y^2 \le a^2$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2+x^2 = x^2+y^2$

$$\int_c P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$ formula Greena

polarne koordinate $x = r\cos\varphi$, $y = r\sin\varphi$ $\Rightarrow$ $D':\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ 0 \le r \le a \\ 0 \le \varphi \le 2\pi \end{cases}$ $dx\,dy = r\,dr\,d\varphi$

$$\int_c xy^2dy - x^2y\,dx = \iint_D (x^2+y^2)\,dx\,dy = \iint_{D'} (r^2\cos^2\varphi + r^2\sin^2\varphi)\cdot r\,dr\,d\varphi =$$

$$= \int_0^{2\pi}\left[\int_0^a r^3dr\right] d\varphi = \int_0^{2\pi} \frac{1}{4}r^4\Big|_0^a d\varphi = \frac{a^4}{4}\cdot\varphi\Big|_0^{2\pi} = \frac{\pi a^4}{2}$$

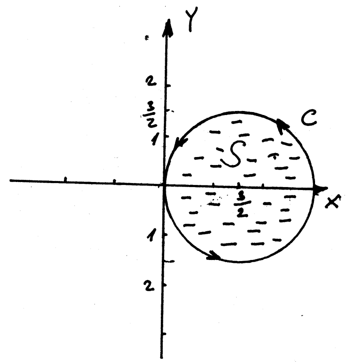
Izračunati krivoliniski integral

$$I=\int_C (xy+x+y)\,dx+(xy+x-y)\,dy \quad \text{ako je } C: x^2+y^2=3x.$$

Rj.

$$x^2+y^2=3x$$
$$x^2-3x+y^2=0$$
$$x^2-2\cdot x\cdot\frac{3}{2}+\frac{9}{4}-\frac{9}{4}+y^2=0$$
$$\left(x-\frac{3}{2}\right)^2+y^2=\frac{9}{4}$$

C: Krug sa centrom u tački $\left(\frac{3}{2},0\right)$ poluprečnika $r=\frac{3}{2}$



I način: Greenova formula za ravan

$$\int_C P\,dx+Q\,dy=\iint_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

c - zatvorena kontura
S - oblast ograničena konturom

$P=xy+x+y$, $\frac{\partial P}{\partial y}=x+1$, $Q=xy+x-y$, $\frac{\partial Q}{\partial x}=y+1$

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=y+1-(x+1)=y-x$$

Kako je c krug, oblast ograničena krugom je unutrašnjost kruga. Da bi smo lakše opisali unutrašnjost kruga uvedimo polarne koordinate $x=\frac{3}{2}+r\cos\varphi$, $y=r\sin\varphi$, $dx\,dy=r\,dr\,d\varphi$

$$D:\begin{cases}0\le r\le \frac{3}{2}\\ 0\le\varphi\le 2\pi\end{cases}$$

$$I=\int_C(xy+x+y)\,dx+(xy+x-y)\,dy=\iint_S(y-x)\,dx\,dy=\iint_D\left(r\sin\varphi-\left(\frac{3}{2}+r\cos\varphi\right)\right)\cdot r\,dr\,d\varphi$$

$$=\int_0^{3/2}\left[\int_0^{2\pi}\left(r^2\sin\varphi-\frac{3}{2}r-r^2\cos\varphi\right)d\varphi\right]dr=\int_0^{3/2}\Big(\underbrace{-r^2\cos\varphi\Big|_0^{2\pi}}_{=0}-\frac{3r}{2}\varphi\Big|_0^{2\pi}\underbrace{-r\sin\varphi\Big|_0^{2\pi}}_{=0}\Big)dr$$

$$=\int_0^{3/2}-3\pi r\,dr=-3\pi\frac{r^2}{2}\Big|_0^{3/2}=-\frac{3}{2}\pi\cdot\frac{9}{4}=-\frac{27}{8}\pi$$

II način: Klasičan način

C kriva u ravni opisana jednačinom $y=\eta(x)$, $a\le x\le b$

$$\int_C P(x,y)\,dx+Q(x,y)\,dy=\int_a^b\left[P(x,\eta(x))+Q(x,\eta(x))\cdot\eta'(x)\right]dx$$

Ako je c data kriva opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$ gdje je $t_1\le t\le t_2$ tada

$$\int_C P(x,y)\,dx+Q(x,y)\,dy=\int_{t_1}^{t_2}\left[P(\mu(t),\eta(t))\mu'(t)+Q(\mu(t),\eta(t))\eta'(t)\right]dt$$

U našem slučaju c je kružnica. Parametrizirajmo kružnicu

$x=\frac{3}{2}+r\cos\varphi$
$y=r\sin\varphi$

U našem slučaju $r=\frac{3}{2}$ a umjesto promjenjive φ stavimo promjenjivu t

$\frac{\partial x}{\partial t}=-\frac{3}{2}\sin t$ $\quad x=\frac{3}{2}+\frac{3}{2}\cos t$

$\frac{\partial y}{\partial t}=\frac{3}{2}\cos t$ $\quad y=\frac{3}{2}\sin t$

gdje $0\le t\le 2\pi$

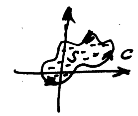
$$I=\int_C(xy+x+y)\,dx+(xy+x-y)\,dy=\int_0^{2\pi}\Big[\Big(\left(\tfrac{3}{2}+\tfrac{3}{2}\cos t\right)\left(\tfrac{3}{2}\sin t\right)+\left(\tfrac{3}{2}+\tfrac{3}{2}\cos t\right)+$$
$$+\left(\tfrac{3}{2}\sin t\right)\Big)\left(-\tfrac{3}{2}\sin t\right)+\Big(\left(\tfrac{3}{2}+\tfrac{3}{2}\cos t\right)\left(\tfrac{3}{2}\sin t\right)+\left(\tfrac{3}{2}+\tfrac{3}{2}\cos t\right)-$$
$$-\left(\tfrac{3}{2}\sin t\right)\Big)\tfrac{3}{2}\cos t\Big]dt=\ldots$$

na klasičan način ovo je komplikovano ali se može izračunati

$$I=-\frac{27}{8}\pi$$

(#) Pomoću Greenove formule izračunati integral $I=\oint\limits_C (xy+x+y)\,dx+(xy+x-y)\,dy$, ako je C kontura kružnice $x^2+y^2=ax$ prijeđena u pozitivnom smislu.

Rj: Greenova formula $\oint\limits_C P\,dx+Q\,dy=\iint\limits_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$



$P(x,y)=xy+x+y \qquad \frac{\partial P}{\partial y}=x+1, \quad \frac{\partial Q}{\partial x}=y+1$

$Q(x,y)=xy+x-y$

$x^2+y^2=ax$

$x^2-ax+y^2=0$

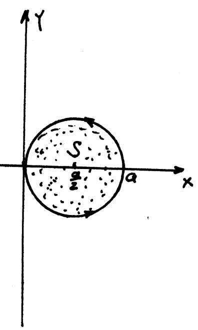
$x^2-2\cdot x\cdot\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}+y^2=0$

$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$

krug sa centrom u $\left(\frac{a}{2},0\right)$ poluprečnika $\frac{a}{2}$

$I=\iint\limits_S (y+1-(x+1))\,dx\,dy$

$I=\iint\limits_S (y-x)\,dx\,dy$



uvedimo polarne koordinate

$x=\frac{a}{2}+r\sin\varphi$

$y=r\cos\varphi$

$dx\,dy=r\,dr\,d\varphi$

$S \xrightarrow{\text{transformiši}} S':\begin{cases}0\le r\le \frac{a}{2}\\ 0\le\varphi\le 2\pi\end{cases}$

$I=\iint\limits_{S'}\left(r\cos\varphi-\frac{a}{2}+r\sin\varphi\right)r\,dr\,d\varphi=\iint\limits_{S'}\left(r^2(\cos\varphi-\sin\varphi)-\frac{a}{2}r\right)dr\,d\varphi=$

$=\int_0^{2\pi}d\varphi\int_0^{\frac{a}{2}}\left[r^2(\cos\varphi-\sin\varphi)-\frac{a}{2}r\right]dr=\int_0^{2\pi}\left[\frac{1}{3}r^3\Big|_0^{\frac{a}{2}}(\cos\varphi-\sin\varphi)-\frac{a}{2}\cdot\frac{1}{2}r^2\Big|_0^{\frac{a}{2}}\right]d\varphi$

$=\frac{a^3}{24}\int_0^{2\pi}(\cos\varphi-\sin\varphi)\,d\varphi-\frac{a^3}{16}\int_0^{2\pi}d\varphi=\frac{a^3}{24}\Big(\underbrace{\sin\varphi\Big|_0^{2\pi}}_{=0}+\underbrace{\cos\varphi\Big|_0^{2\pi}}_{1-1}\Big)-\frac{a^3}{16}2\pi=$

$=-\frac{a^3\pi}{8}$ traženo rješenje

Zadaci za vježbu

U zadacima 3822—3823 krivolinijske integrale po zatvorenim konturama L, uzete u pozitivnom smeru obilaženja, transformisati u dvojne integrale po oblastima, ograničenim tim konturama.

3822. $\int\limits_L (1-x^2)\,y\,dx+x\,(1+y^2)\,dy.$

3823. $\int\limits_L (e^{xy}+2x\cos y)\,dx+(e^{xy}-x^2\sin y)\,dy.$

3824. Izračunati integral u zadatku 3822, ako je kontura integracije L krug $x^2+y^2=R^2$, na dva načina:

1) neposredno;
2) primenom Grinove formule.

3825. Izračunati $\int\limits_L (xy+x+y)\,dx+(xy+x-y)\,dy$, pri čemu je kontura integracije L: 1) elipsa $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; 2) krug $x^2+y^2=ax$, a integral se uzima oba puta u pozitivnom smeru obilaženja. (Račun izvesti na dva načina: 1) neposredno, i 2) primenom Grinove formule).

3826. Dokazati da je integral $\int\limits_L (yx^3+e^y)\,dx+(xy^3+x\,e^y-2\,y)\,dy$ jednak nuli ako je putanja integracije L zatvorena kriva simetrična u odnosu na koordinatni početak.

3827. Primenom Grinove formule izračunati razliku integrala

$$I_1=\int\limits_{AmB}(x+y)^2\,dx-(x-y)^2\,dy$$

i

$$I_2=\int\limits_{AnB}(x+y)^2\,dx-(x-y)^2\,dy,$$

pri čemu je AmB pravolinijski odsečak koji spaja tačke $A\,(0,\,0)$ i $B\,(1,\,1)$, a AnB je luk parabole $y=x^2$.

3828. Pokazati da je vrednost integrala $\int\limits_L \{x\cos(N,\,x)+y\sin(N,\,x)\}\,dS$, u kojem je $(N,\,x)$ ugao između spoljne normale krive L i pozitivnog smera apscisne ose, uzetog u pozitivnom smeru obilaženja po zatvorenoj krivoj L, jednaka dvostrukoj površini oblasti ograničene zatvorenom krivom L.

3829. Dokazati da integral $\int\limits_L (2\,xy-y)\,dx+x^2\,dy$, uzet po zatvorenoj krivoj L, izražava površinu oblasti ograničene tom krivom.

3830. Dokazati da je integral $\int\limits_L \varphi(y)\,dx+[x\varphi'(y)+x^3]\,dy$ jednak trostrukom momentu inercije homogene ravne figure ograničene konturom L, u odnosu na ordinatnu osu.

Rješenja

3822. $\iint\limits_D (x^2+y^2)\,dx\,dy.$

3823. $\iint\limits_D (y-x)\,e^{xy}\,dx\,dy.$

3824. $\frac{\pi R^4}{2}.$

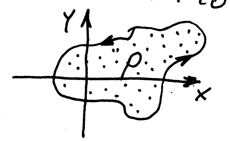
3825. 1) 0; 2) $-\frac{\pi a^3}{8}.$

3827. $\frac{1}{3}.$

Računanje površine ravne figure

Površinu figure ograničenu zatvorenom linijom c računamo po formuli:

$$P=\frac{1}{2}\int_c x\,dy-y\,dx.$$



Podrazumjeva se da po liniji c prelazimo u pozitivnom smjeru.

(#) Pokazati da se površina ograničena jednostrukno zatvorenom krivom (konturom) c računa po formuli

$$\frac{1}{2}\int_c x\,dy-y\,dx$$

Rj. U formuli Greena stavimo $P(x,y)=-y$, $Q(x,y)=x$. Tada

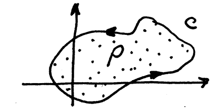
$$\int_c x\,dy-y\,dx=\iint_S\left(\frac{\partial}{\partial x}(x)-\frac{\partial}{\partial y}(-y)\right)dx\,dy=2\iint_S dx\,dy=2\cdot P$$

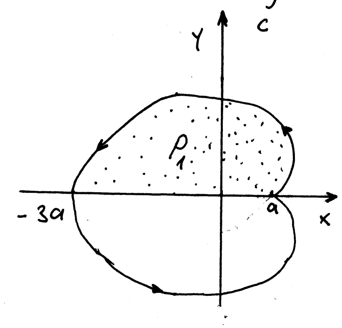
gdje je P tražena površina. Prema tome $P=\frac{1}{2}\int_c x\,dy-y\,dx$

(#) Uz pomoć krivoliniskog integrala druge vrste, izračunati površinu, ograničenu kardioidom $x=2\cos t-\cos 2t$, $y=2\sin t-\sin 2t$.

Rj. Prisjetimo se, površina figure ograničene krivom c se računa po formuli

$$P=\frac{1}{2}\oint_c x\,dy-y\,dx$$





kardioida
$x=2a\cos t-a\cos 2t$
$y=2a\sin t-a\sin 2t$

$t=0$: $x=a$, $y=0$
$t=\pi$: $x=-3a$, $y=0$

Primjetimo da je kardioida kriva linija koja je simetrična u odnosu na x-osu, pa da bi izračunali površinu ograničenu kardioidom dovoljno je izračunati površinu iznad x-ose

Da bismo opisali kardioidu parametar t uzima vrijednosti od 0 do 2π.

Prisjetimo se, ako je kriva c data u parametarskom obliku $x=\mu(t)$, $y=\eta(t)$, $t_1\le t\le t_2$ tada se krivoliniski integral računa po formuli

$$\int_c [P(x,y)\,dx+Q(x,y)\,dy]=\int\left(P(\mu(t),\eta(t))\,\mu'(t)+Q(\mu(t),\eta(t))\cdot\eta'(t)\right)dt$$

$$P=\frac{1}{2}\oint_c x\,dy-y\,dx=\left|\begin{array}{l}x=2\cos t-\cos 2t\\ dx=(-2\sin t+2\sin 2t)dt\\ y=2\sin t-\sin 2t\\ dy=(2\cos t-2\cos 2t)dt\end{array}\right|=\frac{1}{2}\int_0^{2\pi}(2\cos t-\cos 2t)\cdot(2\cos t-2\cos 2t$$

$$=\frac{1}{2}\int_0^{2\pi}\left[(2\cos t-\cos 2t)(2\cos t-2\cos 2t)-(2\sin t-\sin 2t)(-2\sin t+2\sin 2t)\right]dt=2P_1$$

$$=\int_0^{\pi}\left(4\cos^2 t-6\cos t\cos 2t+2\cos^2 2t+4\sin^2 t-6\sin t\sin 2t+2\sin^2 2t\right)dt=$$

$$=\int_0^{\pi}(6-6\cos t\cos 2t-6\sin t\sin 2t)\,dt=6\int_0^{\pi}(1-\cos(t-2t))\,dt=\ldots=6\pi$$

(#) Izračunati pomoću krivolinijskog integrala II vrste površinu ravne figure ograničene konturom

$$c: \begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \\ 0 \le t \le 2\pi \end{cases}$$

Rj: Površina figure ograničenu zatvorenom linijom c računamo po formuli: $P = \frac{1}{2} \int_c x\,dy - y\,dx$.

$x = a(t - \sin t) \qquad y = a(1 - \cos t)$

$dx = a(1 - \cos t) \qquad dy = a \sin t$

$$x\,dy - y\,dx = a(t - \sin t) \cdot a \sin t - a(1 - \cos t) \cdot a(1 - \cos t)$$
$$= a^2 t \sin t - a^2 \sin^2 t - a^2 (1 - \cos t)^2$$
$$= a^2 (t \sin t - \sin^2 t - 1 + 2\cos t - \cos^2 t)$$
$$= a^2 (t \sin t + 2\cos t - 2)$$

$$P = \frac{1}{2} \int_c x\,dy - y\,dx = \frac{1}{2} \int_0^{2\pi} (a^2 (t \sin t + 2\cos t - 2))\,dt =$$

$$= \frac{a^2}{2} \left(\int_0^{2\pi} t \sin t\,dt + 2 \int_0^{2\pi} \cos t\,dt - 2 \int_0^{2\pi} dt \right) = \ldots = \frac{a^2}{2}(-2\pi + 0 - 4\pi) = 3a^2\pi$$

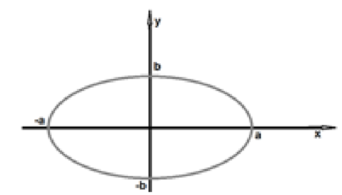
1. Izračunati površinu figure koja je ograničena krivom:

a) elipsom $x = a\cos t, \; y = b\sin t$;

b) petljom Dekartovim listom $x^3 + y^3 - 3axy = 0$.

Rješenja:

a)



Slika 1: elipsa

Koristit ćemo sljedeću formulu:

$$P = \frac{1}{2} \oint_{C_1} x\,dy - y\,dx,$$

gdje je (vidi sliku 1)

$$C_1 = \begin{cases} x = a\cos t \\ y = b\sin t \\ 0 \le t \le 2\pi \end{cases}$$

Izračunajmo izvode od x i y:

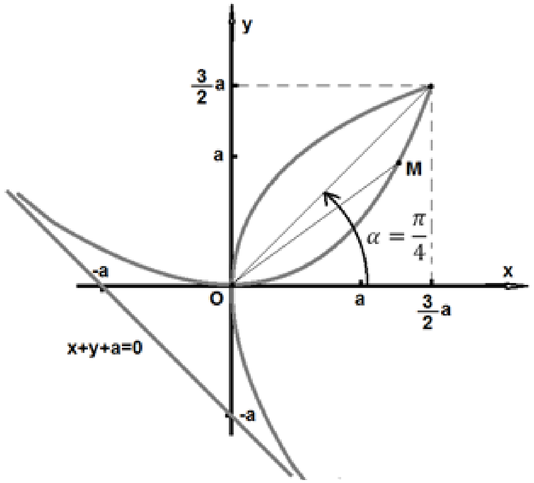
$$dx = -a\sin t\,dt$$
$$dy = b\cos t\,dt$$

Uvrstimo u formulu:

$$P = \frac{1}{2} \oint_{C_1} x\,dy - y\,dx = \frac{1}{2} \int_0^{2\pi} \left(a\cos t\, b\cos t - b\sin t\,(-a)\sin t \right) dt$$
$$= \frac{1}{2} \int_0^{2\pi} (ab\cos^2 t + ab\sin^2 t)\,dt = \frac{1}{2} ab \int_0^{2\pi} \left(\cos^2 t + \sin^2 t \right) dt$$
$$= \frac{1}{2} ab \int_0^{2\pi} 1\,dt = \frac{1}{2} ab \left(t \Big|_0^{2\pi} \right) = \frac{1}{2} ab (2\pi - 0) = \boldsymbol{ab\pi}$$

Konačno rješenje: $\boldsymbol{P = ab\pi}$.

b)



Slika 2: Dekartov list

Da bismo koristili formulu

$$P = \frac{1}{2}\oint_{C_1} x dy - y dx,$$

moramo preći na parametarsku jednačinu krive uzevši:

$$y = tx,\ t = \frac{y}{x}$$

Vidimo da polarni radijus OM (vidi sliku 2), gdje je O(0,0) i M(x,y), opisuje cijelu petlju krive kada t ide od 0 do +∞.

Uvrstimo smjenu $y = tx$ u $x^3 + y^3 - 3axy = 0$ te na dobiveni rezultat unijeti i smjenu $x = \frac{y}{t}$

pa ćemo imati:

$$x^3 + (tx)^3 - 3ax(tx) = 0$$

$$x^3(1+t^3) - 3tax^2 = 0 \ / :x^2$$

$$\frac{x^3(1+t^3) - 3tax^2}{x^2} = 0$$

$$x(1+t^3) - 3ta = 0$$

$$x(1+t^3) = 3ta$$

$$\boldsymbol{x = \frac{3ta}{1+t^3}}$$

$$x = \frac{3ta}{1+t^3}$$

$$\frac{y}{t} = \frac{3ta}{1+t^3}$$

$$\frac{y}{t} = \frac{3ta}{1+t^3}$$

Pa dalje računamo izvod za x:

$$dx = \frac{3a(1+t^3) - 3ta\ (3t^2)}{(1+t^3)^2}dt$$

$$dx = 3a\frac{1+t^3 - t(3t^2)}{(1+t^3)^2}dt$$

$$\boldsymbol{dx = 3a\frac{1-2t^3}{(1+t^3)^2}\ dt}$$

te i za y :

$$dy = \frac{6at(1+t^3) - 3at^2(3t^2)}{(1+t^3)^2}dt$$

$$dy = 3at\frac{2(1+t^3) - t(3t^2)}{(1+t^3)^2}dt$$

$$dy = 3at\frac{2+2t^3-3t^3}{(1+t^3)^2}dt$$

$$\boldsymbol{dy = 3at\frac{2-t^3}{(1+t^3)^2}\ dt}$$

Pomnožimo izvode sa dx i dy sa y i x, redom

$$x\,dy = \frac{3ta}{(1+t^3)}3at\frac{2-t^3}{(1+t^3)^2}dt$$

$$\boldsymbol{x\ dy = 9a^2t^2\frac{2-t^3}{(1+t^3)^3}\ dt}$$

$$y\,dx = \frac{3t^2a}{1+t^3}3a\frac{1-2t^3}{(1+t^3)^2}dt$$

$$\boldsymbol{y\ dx = 9a^2t^2\frac{1-2t^3}{(1+t^3)^3}\ dt}$$

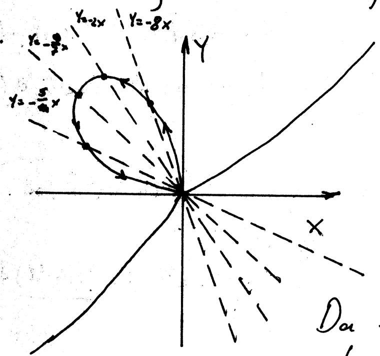
Sad uvrstimo dobijene rezultate:

$$P = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^\infty \left(9a^2t^2\frac{2-t^3}{(1+t^3)^3} - 9a^2t^2\frac{1-2t^3}{(1+t^3)^3}\right)dt$$

$$= \frac{1}{2}\int_0^\infty 9a^2t^2\frac{2-t^3-1+2t^3}{(1+t^3)^3}dt = \frac{9a^2}{2}\int_0^\infty t^2\frac{1+t^3}{(1+t^3)^3}dt =$$

Uz pomoć krivolinijskog integrala izračunati površinu Dekartovog lista dobijen petljom $x^3+y^3-3axy=0$.

Rj.



$$P=\frac{1}{2}\int_C x\,dy-y\,dx$$

Da bismo upotrebili ovu formulu potrebno je parametrizovati krivu.

Da bismo parametrizovali datu petlju, stavimo $y=tx$. Tada iz jednačine krive dobijamo:

$$x^3+y^3-3axy=0$$

$$x^3+t^3x^3-3atx^2=0 \quad /:x^2$$

$$x(1+t^3)=3at$$

$$x=\frac{3at}{1+t^3}$$ (Pokušajte sa slike shvatiti zašto smo stavili $y=tx$!!!)

$$y=tx$$

$$y=\frac{3at^2}{1+t^3}$$

$$dx=3a\,d\left(\frac{t}{1+t^3}\right)=3a\,\frac{1+t^3-t\cdot 3t^2}{(1+t^3)^2}\,dt=3a\,\frac{1-2t^3}{(1+t^3)^2}\,dt$$

$$dy=3a\,d\left(\frac{t^2}{1+t^3}\right)=3a\,\frac{2t(1+t^3)-t^2\cdot 3t^2}{(1+t^3)^2}=3at\,\frac{2+2t^3-3t^3}{(1+t^3)^2}=3at\,\frac{2-t^3}{(1+t^3)^2}$$

$$x\,dy=3at\cdot\frac{1}{1+t^3}\cdot 3at\cdot\frac{2-t^3}{(1+t^3)^2}\,dt=(3at)^2\,\frac{2-t^3}{(1+t^3)^3}\,dt$$

$$y\,dx=3at\,\frac{t}{1+t^3}\cdot 3a\,\frac{1-2t^3}{(1+t^3)^2}\,dt=(3at)^2\,\frac{1-2t^3}{(1+t^3)^3}\,dt$$

$$P=\frac{1}{2}\int_C x\,dy-y\,dx=\frac{1}{2}\int_{-\infty}^{0}9a^2t^2\,\frac{\overbrace{2-t^3-1+2t^3}^{1+t^3}}{(1+t^3)^3}\,dt=\frac{9a^2}{2}\int_{-\infty}^{0}\frac{t^2}{(1+t^3)^2}\,dt=$$

$$=\left|\begin{matrix}1+t^3=u\\ 3t^2dt=du\\ t^2dt=\frac{1}{3}du\end{matrix}\right|=\frac{3a^2}{2}\int_{-\infty}^{0}\frac{du}{u^2}=\frac{3a^2}{2}\cdot\frac{u^{-1}}{-1}=-\frac{3a^2}{2}\cdot\frac{1}{1+t^3}\Bigg|_{-\infty}^{0}$$

$$=-\frac{3a^2}{2}(1-0)=-\frac{3a^2}{2}$$

Površina je uvijek pozitivna $P=\frac{3a^2}{2}$

Izračunati površinu figure koja je ograničena krivom $x=a\cos^3 t$, $y=a\sin^3 t$, $0\le t\le 2\pi$.

Rj.

$$P=\frac{1}{2}\int_C x\,dy-y\,dx,\qquad C:\begin{cases}x=a\cos^3 t\\ y=a\sin^3 t\\ 0\le t\le 2\pi\end{cases}\qquad \begin{matrix}dx=3a\cos^2 t\cdot(-\sin t)\,dt\\ dy=3a\sin^2 t\cos t\,dt\end{matrix}$$

$$P=\frac{1}{2}\int_C x\,dy-y\,dx=\frac{1}{2}\int_0^{2\pi}\left(a\cos^3 t\cdot 3a\sin^2 t\cos t-a\sin^3 t\cdot 3a\cos^2 t\cdot(-\sin t)\right)dt$$

$$=\frac{1}{2}\cdot 3a^2\int_0^{2\pi}(\sin^2 t\cos^4 t+\sin^4 t\cos^2 t)\,dt=\frac{3}{2}a^2\int_0^{2\pi}\sin^2 t\cos^2 t\underbrace{(\cos^2 t+\sin^2 t)}_{1}dt$$

$$=\frac{3}{2}a^2\int_0^{2\pi}\frac{1}{4}\underbrace{(2\sin t\cos t)^2}_{\sin 2t}dt=\frac{3}{8}a^2\int_0^{2\pi}\sin^2 2t\,dt\overset{(*)}{=}\frac{3}{8}a^2\int_0^{2\pi}\frac{1}{2}(1-\cos 4t)\,dt$$

$$=\frac{3}{16}a^2\left(t\Big|_0^{2\pi}-\frac{1}{4}\sin 4t\Big|_0^{2\pi}\right)$$

$$=\frac{3}{16}a^2(2\pi-0)=\frac{3}{8}a^2\pi$$

$1=\sin^2 2t+\cos^2 2t$

$\cos 4t=\cos^2 2t-\sin^2 2t$

$\Rightarrow 1-\cos 4t=2\sin^2 2t \quad \ldots(*)$

Zadaci za vježbu

U zadacima 3861 — 3868 pomoću krivolinijskog integrala izračunati površinu oblasti ograničene datim zatvorenim krivama.

3861. Elipsom $x=a\cos t,\ y=b\sin t$.

3862. Astroidom $x=a\cos^3 t,\ y=a\sin^3 t$.

3863. Kardioidom $x=2a\cos t-a\cos 2t,\ y=2a\sin t-a\sin 2t$.

3864*. Petljom dekartova lista $x^3+y^3-3axy=0$.

3865. Petljom krive $(x+y)^3=xy$.

3866. Petljom krive $(x+y)^4=x^2y$.

3867*. Bernulijevom lemniskatom $(x^2+y^2)^2=2a^2(x^2-y^2)$.

3868. Petljom krive $(\sqrt{x}+\sqrt{y})^{12}=xy$.

Rješenja

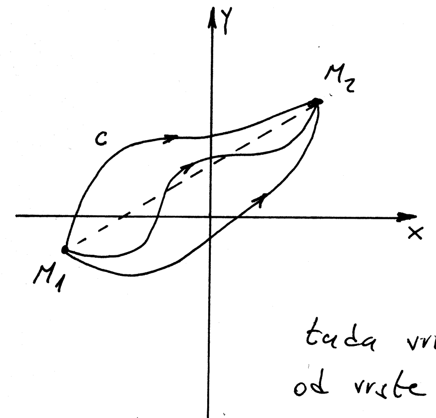
3861. πab. **3862.** $\frac{3}{8}\pi a^2$. **3863.** $6\pi a^2$.

3864*. $\frac{3}{2}a^2$. Preći na parametarske jednačine krive, stavljajući $y=tx$.

3865. $\frac{1}{60}$. **3866.** $\frac{1}{210}$. **3867*.** $2a^2$. Staviti $y=x\operatorname{tg} t$.

3868*. $\frac{1}{30}$. Staviti $y=xt^2$.

Nezavisnost krivolinijskog integrala od vrste krive linije. Određivanje primitivnih f-ja



Ako je data kriva linija c koja spaja tačke $M_1(a,b)$ i $M_2(c,d)$ (pri čemu je M_1 početak a M_2 kraj krive linije c) i krivolinijski integral $I=\int_c P(x,y)\,dx+Q(x,y)\,dy$ kod kojeg vrijedi $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ tada vrijednost integrala I ne zavisi od vrste krive linije c (za krivu liniju c možemo uzeti bilo koju krivu koja spaja tačke M_1 i M_2).

Vrijednost integrala obično tražimo tako što nađemo f-ju $u=u(x,y)$ za koju vrijedi $du(x,y)=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=P(x,y)dx+Q(x,y)dy$ pa imamo

$$I=\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_c du(x,y)=u(x,y)\Big|_{(a,b)}^{(c,d)}=u(c,d)-u(a,b)$$

Izračunati krivolinijski integral $\int_{(0,1)}^{(3,-4)} x\,dx + y\,dy$.

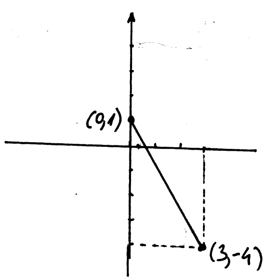
Rj. Integral $I=\int_C P\,dx + Q\,dy$ kod kojeg vrijedi $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, vrijednost integrala I ne zavisi od vrste krive linije C.

U našem slučaju $P(x,y)=x$, $Q(x,y)=y$

$\frac{\partial P}{\partial y}=0$, $\frac{\partial Q}{\partial x}=0$

Prema tome vrijednost ne zavisi od vrste izbora krive linije c koja spaja tačke $(0,1)$ i $(3,-4)$.

I način



Ako je c data kriva u ravni opisana jednačinom $y=\eta(x)$ ($x\in[a,b]$) tada

$$\int_c P\,dx + Q\,dy = \int_a^b [P(x,\eta(x)) + Q(x,\eta(x))\cdot \eta'(x)]\,dx$$

$y-y_1=k(x-x_1)$

$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

$A(0,1)$, $B(3,-4)$

$y-1=\frac{-5}{3}(x-0)$

$y=-\frac{5}{3}x+1$

$y'=-\frac{5}{3}$

$$\int_{(0,1)}^{(3,-4)} x\,dx+y\,dy = \int_0^3 \left(x+\left(-\frac{5}{3}x+1\right)\cdot\left(-\frac{5}{3}\right)\right)dx = \int_0^3 \left(x+\frac{25}{9}x-\frac{5}{3}\right)dx =$$

$$=\left(1+\frac{25}{9}\right)\frac{x^2}{2}\Big|_0^3 - \frac{5}{3}x\Big|_0^3 = \frac{34}{9}\cdot\frac{9}{2}-\frac{5}{3}\cdot 3 = 17-5 = 12$$

II način

$P(x,y)dx+Q(x,y)dy=0$ i $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ je egzaktna diferencijalna jednačina

Rješimo diferenc. jedn. $x\,dx+y\,dy=0$

$u=u(x,y)$

$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$

$\frac{\partial u}{\partial x}=x$, $\frac{\partial u}{\partial y}=y$ ...(1)

$\partial u = x\,\partial x$

$u=\int x\,dx + \varphi(y) = \frac{1}{2}x^2+\varphi(y)$

$\frac{\partial u}{\partial y}=\varphi'(y)$ ...(2)

(1) i (2) $\Rightarrow \varphi'(y)=y$

$\varphi(y)=\frac{1}{2}y^2$

$u=\frac{1}{2}x^2+\frac{1}{2}y^2$

Prema tome $u(x,y)=\frac{1}{2}x^2+\frac{1}{2}y^2$

$$\int_{(0,1)}^{(3,-4)} x\,dx+y\,dy = \int_{(0,1)}^{(3,-4)} du(x,y) = \frac{1}{2}x^2\Big|_{(0,1)}^{(3,-4)} + \frac{1}{2}y^2\Big|_{(0,1)}^{(3,-4)} = \frac{1}{2}(9-0)+\frac{1}{2}(16-1)$$

$$=\frac{9}{2}+\frac{15}{2}=\frac{24}{2}=12$$

Izračunati integral $\displaystyle\int_{(-2,-1)}^{(3,0)} (x^4+4xy^3)\,dx+(6x^2y^2-5y^4)\,dy$

Rj. Označimo sa $P(x,y)=x^4+4xy^3$ i $Q(x,y)=6x^2y^2-5y^4$

$$\frac{\partial P}{\partial y}=12xy^2 \qquad \frac{\partial Q}{\partial x}=12xy^2$$

$P(x,y)\,dx+Q(x,y)\,dy=0$; $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ egzaktna diferencijalna jednačina

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

$$\frac{\partial u}{\partial x}=P(x,y)=x^4+4xy^3 \qquad \frac{\partial u}{\partial y}=Q(x,y)=6x^2y^2-5y^4 \quad \ldots(*)$$

$$\partial u=P(x,y)\,\partial x$$

$$u=\int(x^4+4xy^3)\,dx=\frac{1}{5}x^5+4\cdot\frac{1}{2}x^2y^3+\varphi(y)=\frac{1}{5}x^5+2x^2y^3+\varphi(y)$$

$$\frac{\partial u}{\partial y}=6x^2y^2+\varphi'(y)\ \ldots(**) \qquad \text{Iz } (*) \text{ i } (**) \Rightarrow \varphi'(y)=-5y^4$$

$$\varphi(y)=-5\int y^4\,dy=-y^5$$

Prema tome $u(x,y)=\frac{1}{5}x^5+2x^2y^3-y^5$

$$\int_{(-3,-1)}^{(3,0)}(x^4+4xy^3)\,dx+(6x^2y^5-5y^4)\,dy=\int_{(-2,1)}^{(3,0)}du(x,y)=\left(\frac{1}{5}x^5+2x^2y^3-y^5\right)\Bigg|_{(-3,1)}^{(3,0)}$$

$$=\left(\frac{3^5}{5}+0+0\right)-\left(\frac{(-2)^5}{5}+2\cdot4-1\right)=\frac{243}{5}+\frac{32}{5}-\frac{40}{5}+\frac{5}{5}=\frac{240}{5}=48$$

Dokazati da integral $\oint_L f(xy)(y\,dx+x\,dy)$ po zatvorenoj konturi L ima vrijednost 0 (nula) bez obzira na tip f-je uključen u integrand.

Rj. $$\int_L f(xy)(y\,dx+x\,dy)=\int_L y f(xy)\,dx+x f(xy)\,dy$$

Označimo sa $P(x,y)=y f(xy)$ i $Q(x,y)=x f(xy)$. Imamo

$$\frac{\partial P}{\partial y}=f(xy)+y\cdot\frac{\partial f}{\partial(xy)}\cdot x=f(xy)+xy\cdot\frac{\partial f}{\partial(xy)}$$

$$\frac{\partial Q}{\partial x}=f(xy)+x\frac{\partial f}{\partial(xy)}\cdot y=f(xy)+xy\frac{\partial f}{\partial(xy)}$$

$$\Rightarrow \frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

$$\int_C P\,dx+Q\,dy=\iint_S\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)dx\,dy \quad \text{formula Greena}$$

$$\int_L f(xy)(y\,dx+x\,dy)=\iint_S\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)dx\,dy=0 \quad \text{bez obzira na } L.$$

š.e.d.

\# Izračunati krivolinijski integral $\int_C \cos 2y\,dx - 2x\sin 2y\,dy$

gdje je c neka kriva koja spaja tačke $A(1,\frac{\pi}{6})$ i $B(2,\frac{\pi}{4})$.

Rj. Označimo sa $P(x,y)=\cos 2y$ i $Q(x,y)=(-2x)\sin y$

$\frac{\partial P}{\partial y} = -2\sin 2y \qquad \frac{\partial Q}{\partial x} = -2\sin y \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

vrijednost integrala ne zavisi od vrste konture

I način:

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C du(x,y) = u(x,y)\Big|_{(a,b)}^{(c,d)}$$ gdje je

$du(x,y) = P(x,y)\,dx + Q(x,y)\,dy$, tačka (a,b) početak a (c,d) kraj konture c

Odredimo f-ju $u=u(x,y)$

$\frac{\partial u}{\partial x} = P(x,y) = \cos 2y, \quad \frac{\partial u}{\partial y} = Q(x,y) = -2x\sin 2y$

$P(x,y)\,dx + Q(x,y)\,dy = 0$; $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ovo je egzaktna diferencijalna jednačina

$\frac{\partial u}{\partial x} = \cos 2y$

$\partial u = \cos 2y\,\partial x \qquad \frac{\partial u}{\partial y} = -2x\sin 2y \quad \ldots(*)$

$u = \int \cos 2y\,dx = x\cos 2y + \varphi(y) \Rightarrow \frac{\partial u}{\partial y} = x\cdot(-\sin 2y)\cdot 2 + \varphi'(y) = -2x\sin 2y + \varphi'(y) \quad \ldots(**)$

Sad imamo (iz (*) i (**)) $\varphi'(y) = 0 \Rightarrow \varphi(y) = c$

$u(x,y) = x\cos 2y + c$

$$\int_C \cos 2y\,dx - 2x\sin 2y\,dy = \int_C d(x\cos 2y + c) = x\cos 2y\Big|_{(1,\frac{\pi}{6})}^{(2,\frac{\pi}{4})} + c\Big|_{(1,\frac{\pi}{6})}^{(2,\frac{\pi}{4})} =$$

$$= 2\cos\frac{\pi}{2} - \cos\frac{\pi}{3} + (c-c) = -\frac{1}{2}$$

II način: (za vježbu) standardno rješavamo krivolinijski integral s tim da izaberemo pogodnu konturu koja spaja date tačke

\# Izračunati integral po glatkom luku koji spaja tačke A i B

$$\int_{\widehat{AB}} \left(1-\frac{1}{y}+\frac{y}{z}\right)dx + \left(\frac{x}{z}+\frac{x}{y^2}\right)dy - \frac{xy}{z^2}\,dz$$

$A(1,1,1)$, $B(1,2,3)$, $\widehat{AB} \subseteq \{(x,y,z) \mid x>0,\ y>0,\ z>0\}$.

Rj. Označimo sa $P(x,y,z) = 1-\frac{1}{y}+\frac{y}{z}$, $Q(x,y,z) = \frac{x}{z}+\frac{x}{y^2}$, $R(x,y,z) = -\frac{xy}{z^2}$, i izračunajmo $\frac{\partial^2 P}{\partial y\partial z}$, $\frac{\partial^2 Q}{\partial x\partial z}$ i $\frac{\partial^2 R}{\partial x\partial y}$

$\frac{\partial P}{\partial y} = -(-1)y^{-2} + \frac{1}{z} \qquad \frac{\partial Q}{\partial x} = \frac{1}{z} + \frac{1}{y^2} \qquad \frac{\partial R}{\partial x} = -\frac{y}{z^2}$

$\frac{\partial^2 P}{\partial y\partial z} = -\frac{1}{z^2} \qquad \frac{\partial^2 Q}{\partial x\partial z} = -\frac{1}{z^2} \qquad \frac{\partial^2 R}{\partial x\partial y} = -\frac{1}{z^2}$

Kako je $\frac{\partial^2 P}{\partial y\partial z} = \frac{\partial^2 Q}{\partial x\partial z} = \frac{\partial^2 R}{\partial x\partial y}$ to integral ne zavisi od vrste krive linije koja spaja tačke A i B.

Odredimo f-ju $u=u(x,y,z)$ za koju vrijedi da je

$$du = \left(1-\frac{1}{y}+\frac{y}{z}\right)dx + \left(\frac{x}{z}+\frac{x}{y^2}\right)dy - \frac{xy}{z^2}\,dz$$

$\frac{\partial u}{\partial x} = 1-\frac{1}{y}+\frac{y}{z}$;

$u = \int\left(1-\frac{1}{y}+\frac{y}{z}\right)dx + \varphi(y,z)$

$u = x - \frac{x}{y} + \frac{xy}{z} + \varphi(y,z)$

$\frac{\partial u}{\partial y} = \frac{x}{y^2} + \frac{x}{z} + \varphi'_y(y,z)$

$\frac{\partial u}{\partial y} = \frac{x}{y^2} + \frac{x}{z}$

$\varphi'_y(y,z) = 0$

$\varphi(y,z) = C + \psi(z) \quad \ldots(1)$

$\frac{\partial u}{\partial z} = -\frac{xy}{z^2} + \varphi'_z$

$\frac{\partial u}{\partial z} = -\frac{xy}{z^2}$

$\varphi'_z = 0 \quad \ldots(2)$

(1) i (2) $\Rightarrow \psi(z) = 0 \Rightarrow \varphi(y,z) = C$

$u = x - \frac{x}{y} + \frac{xy}{z} + C$

$$\int_{\widehat{AB}} \left(1-\frac{1}{y}+\frac{y}{z}\right)dx + \left(\frac{x}{z}+\frac{x}{y^2}\right)dy - \frac{xy}{z^2}\,dz = \int_{\widehat{AB}} du = \left(x - \frac{x}{y} + \frac{xy}{z}\right)\Big|_{(1,1,1)}^{(1,2,3)} = 1 - \frac{1}{2} + \frac{2}{3} - 1 = \frac{1}{6}$$

tražena vrijednost

Izračunati krivoliniski integral $\int_{(2,1)}^{(1,2)} \frac{y\,dx - x\,dy}{x^2}$ duž putanje koja ne siječe osu Oy.

Rj:

Vrijednost integrala $I = \int_c P(x,y)\,dx + Q(x,y)\,dy$ ne zavisi od vrste konture c ako je $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

U našem slučaju $I = \int_{(2,1)}^{(1,2)} \frac{y}{x^2}\,dx - \frac{1}{x}\,dy$

$P(x,y) = \frac{y}{x^2}$, $Q(x,y) = -\frac{1}{x}$

$\frac{\partial P}{\partial y} = \frac{1}{x^2}$, $\frac{\partial Q}{\partial x} = \frac{1}{x^2}$

Prema tome vrijednost integrala ne zavisi od vrste krive linije c koja spaja tačke (2,1) i (1,2).

I način: Odredićemo primitivnu f-ju

$P(x,y)\,dx + Q(x,y)\,dy = 0$ — ovo je egzaktna dif. jednačina

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$u = u(x,y)$

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$du = \frac{y}{x^2}dx - \frac{1}{x}dy$

$\frac{\partial u}{\partial x} = \frac{y}{x^2}$, $\frac{\partial u}{\partial y} = -\frac{1}{x}$ ...(1)

$\partial u = \frac{y}{x^2}\partial x$

$u = \int \frac{y}{x^2}dx + \varphi(y) = y\frac{x^{-1}}{-1} + \varphi(y) = -\frac{y}{x} + \varphi(y)$

$\frac{\partial u}{\partial y} = -\frac{1}{x} + \varphi'(y)$ ...(2)

(1) i (2) $\Rightarrow \varphi'(y) = 0$, $\varphi(y) = c$

$u = -\frac{y}{x} + c$

$$\int_{(2,1)}^{(1,2)} \frac{y\,dx - x\,dy}{x^2} = \int_{(2,1)}^{(1,2)} du = -\frac{y}{x}\Big|_{(2,1)}^{(1,2)} = -\frac{2}{1} - \left(-\frac{1}{2}\right) = \frac{1}{2} - 2 = -\frac{3}{2}$$

II način: Spojimo tačke (2,1) i (1,2) nekom krivom (ili pravom) ili izlomljenom pravom linijom i izračunamo integral na klasičan način.

Izračunati krivoliniski integral $\int_{(1,0)}^{(6,8)} \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$ duž puta koji ne prolazi kroz koordinatni početak.

Rj: Ako je $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ tada vrijednost integrala $\int_c P\,dx + Q\,dy$ ne zavisi od vrste izbora puta integracije.

$I = \int_{(1,0)}^{(6,8)} \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy \Rightarrow P(x,y) = \frac{x}{\sqrt{x^2+y^2}}$, $Q(x,y) = \frac{y}{\sqrt{x^2+y^2}}$ $\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -\frac{xy}{\sqrt{(x^2+y^2)^3}}$

Prema tome vrijednost integrala ne zavisi od izbora krive kojom ćemo spojiti tačke (1,0) i (6,8).

I način: Odrediti ćemo primitivnu f-ju u.

$u = u(x,y)$

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$du = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy$

$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$ ...(1)

$\partial u = \frac{x}{\sqrt{x^2+y^2}}\partial x$

$u = \int \frac{x}{\sqrt{x^2+y^2}}dx + \varphi(y) = \left| \begin{array}{l} x^2+y^2 = t^2 \\ 2x\,dx = 2t\,dt \\ x\,dx = t\,dt \end{array} \right| = \int \frac{t}{\sqrt{t^2}}dt + \varphi(y) = t + \varphi(y) = \sqrt{x^2+y^2} + \varphi(y)$

$\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} + \varphi'(y)$ ...(2)

(1) i (2) $\Rightarrow \varphi'(y) = 0 \Rightarrow u = \sqrt{x^2+y^2}$

$$\int_{(1,0)}^{(6,8)} \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = \int_{(1,0)}^{(6,8)} du = u\Big|_{(1,0)}^{(6,8)} = \sqrt{x^2+y^2}\Big|_{(1,0)}^{(6,8)} = \sqrt{36+64} - \sqrt{1+0} = 9$$

II način: Spojimo tačke (1,0) i (6,8) nekom krivom koja ne prolazi kroz koordinatni početak i izračunamo integral na klasičan način.

Zadaci za vježbu

U zadacima 3831 — 3835 uveriti se da su vrednosti datih integrala, uzetih po zatvorenim konturama, jednake nuli bez obzira na oblik funkcija koje ulaze u podintegralni izraz.

3831. $\int\limits_L \varphi(x)\,dx + \psi(y)\,dy$. **3832.** $\int\limits_L f(xy)\,(y\,dx + x\,dy)$.

3833. $\int\limits_L f\left(\frac{y}{x}\right)\frac{x\,dy - y\,dx}{x^2}$.

3834. $\int\limits_L [f(x+y) + f(x-y)]\,dx + [f(x+y) - f(x-y)]\,dy$.

3835. $\int\limits_L f(x^2+y^2+z^2)\,(x\,dx + y\,dy + z\,dz)$.

3836*. Dokazati da integral $\int\limits_L \frac{x\,dy - y\,dx}{x+y}$, uzet u pozitivnom smeru obilaženja po bilo kojoj zatvorenoj konturi koja obuhvata koordinatni početak, ima vrednost 2π.

3837. Izračunati $\int\limits_L \frac{x\,dy - y\,dx}{x^2+4y^2}$ duž kruga $x^2+y^2=1$ u pozitivnom smeru obilaženja.

U zadacima 3838—3844 izračunati krivolinijske integrale totalnih diferencijala.

3838. $\int\limits_{(-1,2)}^{(2,3)} y\,dx + x\,dy$. **3839.** $\int\limits_{(0,0)}^{(2,1)} 2xy\,dx + x^2\,dy$.

3840. $\int\limits_{(3,4)}^{(5,12)} \frac{x\,dx + y\,dy}{x^2+y^2}$ (koordinatni početak ne leži na putanji integracije).

3841. $\int\limits_{(P_1)}^{(P_2)} \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$, pri čemu tačke P_1 i P_2 leže na koncentričnim krugovima čiji je zajednički centar u koordinatnom početku, a poluprcčnici su im R_1 i R_2 (koordinatni početak ne leži na putanji integracije).

3842. $\int\limits_{(1,-1,2)}^{(2,1,3)} x\,dx - y^2\,dy + z\,dz$.

3843. $\int\limits_{(1,2,3)}^{(3,2,1)} yz\,dx + zx\,dy + xy\,dz$.

3844. $\int\limits_{(7,2,3)}^{(5,3,1)} \frac{zx\,dy + xy\,dz - yz\,dx}{(x-yz)^2}$ (putanja integracije ne preseca površinu $z=\frac{x}{y}$).

U zadacima 3845—3852 naći funkcije čiji su totalni diferencijali zadati.

3845. $du = x^2\,dx + y^2\,dy$.

3846. $du = 4(x^2-y^2)(x\,dx - y\,dy)$.

3847. $du = \frac{(x+2y)\,dx + y\,dy}{(x+y)^2}$.

3848. $du = \frac{x}{y\sqrt{x^2+y^2}}\,dx - \left(\frac{x^2+\sqrt{x^2+y^2}}{y^2\sqrt{x^2+y^2}}\right)dy$.

3849. $du = \left[\frac{x-2y}{(y-x)^2} + x\right]dx + \left[\frac{y}{(y-x)^2} - y^2\right]dy$.

3850. $du = (2x\cos y - y^2\sin x)\,dx + (2y\cos x - x^2\sin y)\,dy$.

3851. $du = \frac{2x(1-e^y)}{(1+x^2)^2}\,dx + \left(\frac{e^y}{1+x^2} + 1\right)dy$.

Rješenja

3836*. Primeniti Grinovu formulu na dvostruko povezanu oblast, ograničenu zatvorenom konturom L i bilo kakvim krugom čiji je centar u koordinatnom početku i koji ne preseca konturu L.

3837. π. **3838.** 8.

3839. 4. **3840.** $\ln\frac{13}{5}$.

3841. $R_2 - R_1$. **3842.** $\frac{10}{3}$.

3843. 0. **3844.** $-\frac{9}{2}$.

3845. $u = \frac{x^3+y^3}{3} + C$.

3846. $u = (x^2-y^2)^2 + C$.

3847. $u = \ln|x+y| - \frac{y}{x+y} + C$.

3848. $u = \frac{\sqrt{x^2+y^2}+1}{y} + C$.

3849. $u = \ln|x-y| + \frac{y}{x-y} + \frac{x^2}{2} - \frac{y^3}{3} + C$.

3850. $u = x^2\cos y + y^2\cos x + C$.

3851. $u = \frac{e^y-1}{1+x^2} + y + C$.

3852. $du = \frac{(3y-x)\,dx + (y-3x)\,dy}{(x+y)^3}$.

3853. Odrediti broj n tako da izraz $\frac{(x-y)\,dx + (x+y)\,dy}{(x^2+y^2)^n}$ bude totalni diferencijal, i naći odgovarajuću primitivnu funkciju.

3854. Odrediti konstante a i b tako da izraz

$$\frac{(y^2+2xy+ax^2)\,dx - (x^2+2xy+by^2)\,dy}{(x^2+y^2)^2}$$

bude totalan diferencijal, i naći odgovarajuću primitivnu funkciju.

U zadacima 3855 — 3860 naći funkcije čiji su totalni diferencijali zadati.

3855. $du = \frac{dx+dy+dz}{x+y+z}$. **3856.** $du = \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}$.

3857. $du = \frac{yz\,dx + xz\,dy + xy\,dz}{1+x^2y^2z^2}$.

3858. $du = \frac{2(zx\,dy + xy\,dz - yz\,dx)}{(x-yz)^2}$.

3859. $du = \frac{dx - 3\,dy}{z} + \frac{3y - x + z^3}{z^2}\,dz$.

3860. $du = e^{\frac{y}{z}}\,dx + \left(\frac{e^{\frac{y}{z}}(x+1)}{z} + ze^{yz}\right)dy + \left(-\frac{e^{\frac{y}{z}}(x+1)y}{z^2} + ye^{yz} + e^{-z}\right)dz$.

Rješenja

3852. $u = \frac{x-y}{(x+y)^2} + C$. **3853.** $n=1$, $u = \frac{1}{2}\ln(x^2+y^2) + \operatorname{arctg}\frac{y}{x} + C$.

3854. $a=b=-1$, $u = \frac{x-y}{x^2+y^2} + C$. **3855.** $u = \ln|x+y+z| + C$.

3856. $u = \sqrt{x^2+y^2+z^2} + C$. **3857.** $u = \operatorname{arctg} xyz + C$.

3858. $u = \frac{2x}{x-yz} + C$. **3859.** $u = \frac{x-3y}{z} + \frac{z^2}{2} + C$.

3860. $u = e^{\frac{y}{z}}(x+1) + e^{yz} - e^{-z}$.

Površinski integral prve vrste

Trebamo izračunati integral $\iint\limits_S f(x,y,z)\,dS$ gdje je S - površ u prostoru.

I način:

Ako je D projekcija površi $S: z=z(x,y)$ na xOy ravan tada

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f(x,y,z(x,y))\cdot\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

II način:

L je projekcija površi $S: y=y(x,z)$ na xOz ravan

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_L f(x,y(x,z),z)\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dx\,dz$$

III način:

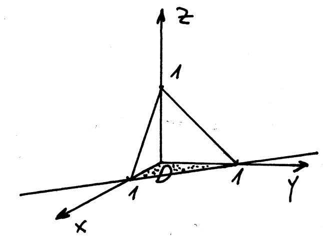
Neka je C projekcija površi $S: x=x(y,z)$ na yOz ravan

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_C f(x(y,z),y,z)\sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\,dy\,dz$$

(#) Izračunati površinski integral $I=\iint\limits_S xyz\,dS$, ako je S dio ravni $x+y+z=1$ u I oktantu.

Rj.

$x+y+z=1$ je ravan koja na x, y i z osi odsjeca 1.

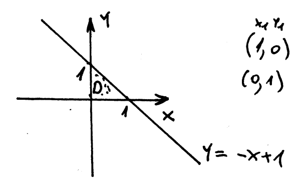


Ako je S data površ opisana jednačinom $z=z(x,y)$ i ako je D projekcija površi S na xOy ravan tada:

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f(x,y,z(x,y))\cdot\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

U našem slučaju $z=1-x-y$, $\frac{\partial z}{\partial x}=-1$, $\frac{\partial z}{\partial y}=-1$

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+1+1} = \sqrt{3}$$



$x_1\ y_1$

$(1,0)$ $\quad y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

$(0,1)$

$y=-x+1$

$$D:\begin{cases} 0\le x\le 1\\ 0\le y\le -x+1\end{cases}$$

1
11
121
1331

Sad imamo

$$I=\iint\limits_S xyz\,dS = \sqrt{3}\iint\limits_D x\cdot y\cdot(1-x-y)\,dx\,dy = \sqrt{3}\int_0^1 x\,dx\int_0^{-x+1}(y-xy-y^2)\,dy =$$

$$=\sqrt{3}\int_0^1 x\left(\frac{1}{2}y^2\Big|_0^{-x+1} - x\cdot\frac{1}{2}y^2\Big|_0^{(-x+1)} - \frac{1}{3}y^3\Big|_0^{(-x+1)}\right)dx =$$

$$=\sqrt{3}\int_0^1\left(\frac{1}{2}x\overbrace{(-x+1)^2}^{x^2-2x+1} - \frac{1}{2}x^2\overbrace{(-x+1)^2}^{x^2-2x+1} - \frac{1}{3}x\overbrace{(-x+1)^3}^{-x^3+3x^2-3x+1}\right)dx =$$

$$=\sqrt{3}\int_0^1\left(\frac{1}{2}x^3-x^2+\frac{1}{2}x-\frac{1}{2}x^4+x^3-\frac{1}{2}x^2+\frac{1}{3}x^4-x^3+x^2-\frac{1}{3}x\right)dx$$

$$=\sqrt{3}\int_0^1\left(-\frac{1}{6}x^4+\frac{1}{2}x^3-\frac{1}{2}x^2+\frac{1}{6}x\right)dx = \sqrt{3}\underbrace{\left(-\frac{1}{6}\cdot\frac{1}{5}+\frac{1}{2}\cdot\frac{1}{4}-\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{6}\cdot\frac{1}{2}\right)}_{\frac{1}{120}} = \frac{\sqrt{3}}{120}$$

rješenje

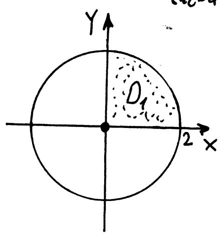
Izračunati površinski integral $\iint_{(S)} \sqrt{-x^2+4}\, dS$, gde je (S) omotač površi $\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$, $0 \le z \le 3$.

Rj. Skicirajmo površ $\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$, $0 \le z \le 3$

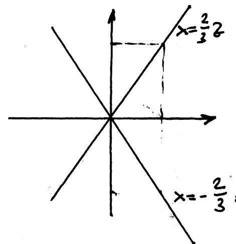
u xOy ravni

$$\frac{x^2}{4}+\frac{y^2}{4}=0$$

za $z=0$, $x^2+y^2=0$ tačka $(0,0)$



u xOz ravni



$$\frac{x^2}{4}=\frac{z^2}{9}$$

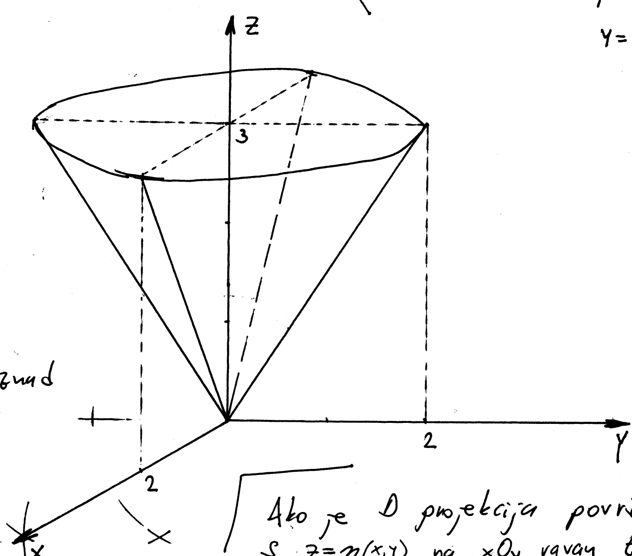
$$x^2=\frac{4}{9}z^2$$

$$x=\pm\frac{2}{3}z$$

yOz ravan

$$y=\pm\frac{2}{3}z$$

za $z=3$ $x^2+y^2=4$



$$\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$$

$$z^2=\frac{9}{4}(x^2+y^2)$$

Kako je data površ iznad xOy ravni

$$z=\frac{3}{2}\sqrt{x^2+y^2}$$

$$z'_x=\frac{3}{2}\cdot\frac{2x}{2\sqrt{x^2+y^2}}=\frac{3x}{2\sqrt{x^2+y^2}} \qquad z'_y=\frac{3y}{2\sqrt{x^2+y^2}}$$

Ako je D projekcija površi S $z=\eta(x,y)$ na xOy ravan tada

$$\iint_S f(x,y,z)\,dS=\iint_D f(x,y,\eta(x,y))\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

$$1+(z'_x)^2+(z'_y)^2=1+\frac{9x^2}{4(x^2+y^2)}+\frac{9y^2}{4(x^2+y^2)}=\frac{13x^2+13y^2}{4(x^2+y^2)}=\frac{13}{4}$$

Primetimo da je data površ (S) simetrična u odnosu na xOz ravan i yOz ravan pa možemo pisati

$$\iint_{(S)}\sqrt{-x^2+4}\,dS=\frac{\sqrt{13}}{2}\iint_D\sqrt{-x^2+4}\,dx\,dy=4\cdot\frac{\sqrt{13}}{2}\iint_{D_1}\sqrt{4-x^2}\,dx\,dy$$

gde je $D_1:\begin{cases}0\le x\le 2\\ 0\le y\le\sqrt{4-x^2}\end{cases}$

$$\iint_{(S)}\sqrt{-x^2+4}\,dS=2\sqrt{13}\int_0^2\sqrt{4-x^2}\,dx\int_0^{\sqrt{4-x^2}}dy=2\sqrt{13}\int_0^2(4-x^2)\,dx=$$

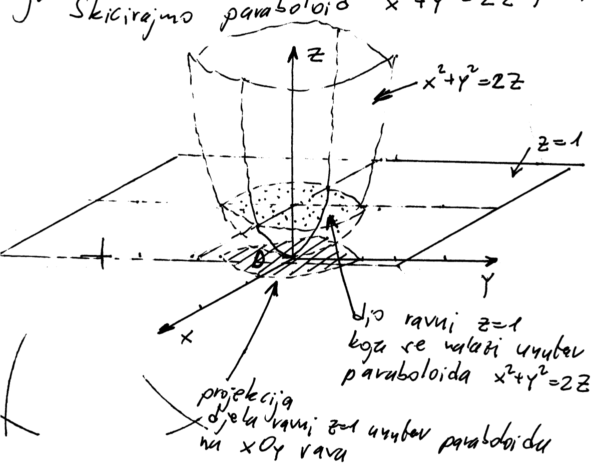
$$=2\sqrt{13}\left(4x\Big|_0^2-\frac{1}{3}x^3\Big|_0^2\right)=2\sqrt{13}\left(8-\frac{8}{3}\right)=2\sqrt{13}\cdot\frac{24-8}{3}$$

$$=\frac{32}{3}\sqrt{13} \quad \text{traženo rešenje}$$

\# Izračunati površinski integral prvog tipa

$$\iint_W (x^2+y^2)\,dS,$$ gdje je W - površina dijela ravni $z=1$ koja se nalazi unutar paraboloida $x^2+y^2=2z$.

Rj. Skicirajmo paraboloid $x^2+y^2=2z$ i ravan $z=1$.



Prisjetimo se kako se računa površinski integral prvog tipa

$$\iint_W f(x,y,z)\,dS = \iint_D f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

gdje je D projekcija površi W na xOy ravan, a W je opisana formulom $z=z(x,y)$.

Projekcija površi W na xOy ravan u našem slučaju je D: unutrašnjost kruga $x^2+y^2=2$.

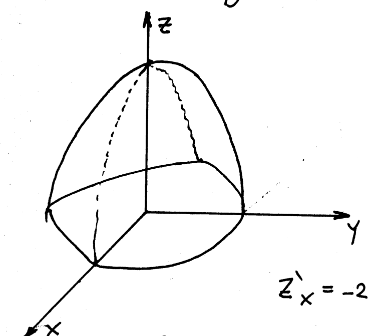
$W: z=1 \Rightarrow \frac{\partial z}{\partial x}=0$ i $\frac{\partial z}{\partial y}=0$

$$\iint_W (x^2+y^2)\,dS = \iint_D (x^2+y^2)\sqrt{1+0+0}\,dx\,dy = \left| \begin{array}{l} \text{uvedimo polarne koordinate} \\ x=r\cos\varphi \\ y=r\sin\varphi \\ dx\,dy=r\,dr\,d\varphi \\ x^2+y^2=r^2 \end{array} \quad D \xrightarrow{\text{transf.}} D': \begin{cases} 0\le r\le\sqrt{2} \\ 0\le\varphi\le 2\pi \end{cases} \right|$$

$$= \iint_{D'} r^2 r\,dr\,d\varphi = \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}} r^3\,dr = \varphi\Big|_0^{2\pi} \cdot \frac{1}{4} r^4\Big|_0^{\sqrt{2}} = 2\pi \cdot \frac{1}{4} \cdot 4 = 2\pi$$

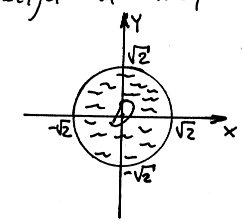
\# Izračunati $\iint_S U(x,y,z)\,dS$ gdje je S površina paraboloida $z=2-(x^2+y^2)$ iznad xy ravni i $U(x,y,z)$ je jednako a) 1 b) x^2+y^2 c) $3z$.

Rj. $$\iint_S U(x,y,z)\,dS = \iint_D U(x,y,z)\sqrt{1+{z'_x}^2+{z'_y}^2}\,dx\,dy$$ gdje je oblast D projekcija površi S na xOy ravan



$z=2-(x^2+y^2)$

Projekcija na xOy ravan $x^2+y^2=2$



$z'_x=-2x$

$z'_y=-2y$

$$\iint_S U(x,y,z)\,dS = \iint_D U(x,y,z)\sqrt{1+4x^2+4y^2}\,dx\,dy$$

a) $U(x,y,z)=1$ $\qquad I=\iint_D \sqrt{1+4x^2+4y^2}\,dx\,dy$

Da izračunamo ovo transformišimo na polarne koordinate $x=r\cos\varphi$, $y=r\sin\varphi$

$$D': \begin{cases} x=r\cos\varphi \\ y=r\sin\varphi \\ 0\le\varphi\le 2\pi \\ 0\le r\le\sqrt{2} \end{cases} \qquad dx\,dy=r\,d\varphi\,dr$$

$$I=\iint_{D'}\sqrt{1+4r^2}\cdot r\,dr\,d\varphi =$$

$$= \int_0^{2\pi}\left[\int_0^{\sqrt{2}}\sqrt{1+4r^2}\cdot r\,dr\right]d\varphi = \left| \begin{array}{ll} 1+4r^2=t^2 & r=0 \Rightarrow t=1 \\ 8r\,dr=2t\,dt & r=\sqrt{2} \Rightarrow t=3 \\ r\,dr=\frac{1}{4}t\,dt & \end{array} \right| =$$

$$= \int_0^{2\pi}\left[\int_1^3 t\cdot\frac{1}{4}t\,dt\right]d\varphi = \frac{1}{4}\int_0^{2\pi}\frac{1}{3}t^3\Big|_1^3 d\varphi = \frac{1}{12}\cdot\varphi\Big|_0^{2\pi}\cdot 26 = \frac{13}{6}\cdot 2\pi = \frac{13\pi}{3}$$

b) vježbu $$I=\iint_D (x^2+y^2)\sqrt{1+4x^2+4y^2} = \iint_{D'} r^3\sqrt{1+4r^2}\,dr\,d\varphi = \frac{149}{30}$$

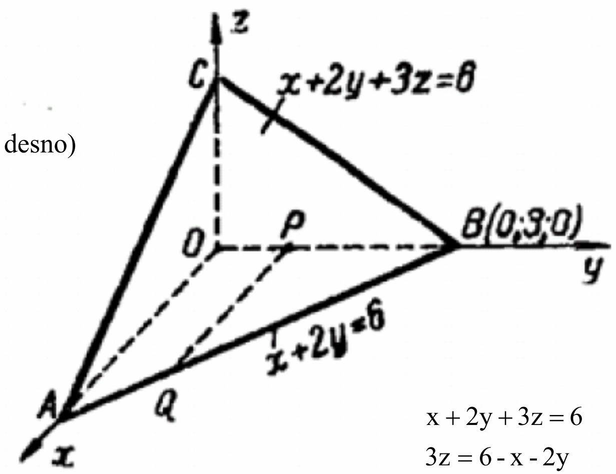
c) vjež. $$I=\frac{111\pi}{10}$$

1. Izračunati površinski integral:

a) $I = \iint_{\sigma} (6x + 4y + 3z)ds$, gdje je σ oblast ravni x+2y+3z = 6, u prvom oktantu;

b) $K = \iint_{W} (y + z + \sqrt{a^2 - x^2})ds$, gdje je W površina cilindra $x^2 + y^2 = a^2$, koja se nalazi između ravni z = 0 i z = h.

Rješenja:



a) Skicirajmo oblast σ (vidi sliku desno)

x+2y+3z = 6/:6

$$\frac{x}{6} + \frac{y}{3} + \frac{z}{2} = 1$$

segmentni oblik jednačine ravni

$$x + 2y + 3z = 6$$
$$3z = 6 - x - 2y$$
$$z = \frac{1}{3}(6 - x - 2y)$$

$$\iint_{\sigma} f(x,y,z)ds = \iint_{D} f(x,y,z(x,y)) \cdot \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy$$

$$\frac{\partial z}{\partial x} = -\frac{1}{3}$$
$$\frac{\partial z}{\partial y} = -\frac{2}{3}$$

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{1}{9} + \frac{4}{9}} = \sqrt{\frac{14}{9}} = \frac{\sqrt{14}}{3}$$

Projekcija na xOy ravan izgleda: Nacrtati projekciju (uputa:vidi xOy ravan sa slike iznad).

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$
$$\frac{x - 6}{0 - 6} = \frac{y - 0}{3 - 0}$$
$$\frac{x - 6}{-6} = \frac{y}{3}$$
$$3x - 18 = -6y$$
$$3x = 18 - 6y$$
$$x = 6 - 2y$$

$$D: \begin{cases} 0 \le y \le 3 \\ 0 \le x \le 6 - 2y \end{cases}$$

$$I = \iint_{\sigma}(6x + 4y + 3z)ds = \frac{\sqrt{14}}{3}\iint_{D}(6x + 4y + 6 - x - 2y)dxdy = \frac{\sqrt{14}}{3}\iint_{D}(5x + 2y + 6)dxdy =$$

$$\frac{\sqrt{14}}{3}\int_0^3 dy \int_0^{6-2y}(5x + 2y + 6)dx = \frac{\sqrt{14}}{3}\int_0^3\left(\frac{5}{2}x^2\Big|_0^{6-2y} + 2xy\Big|_0^{6-2y} + 6x\Big|_0^{6-2y}\right)dy =$$

$$= \frac{\sqrt{14}}{3}\int_0^3\left(\frac{5}{2}(6 - 2y)^2 + 2\cdot(6 - 2y)\cdot y + 6\cdot(6 - 2y)\right)dy =$$

$$= \frac{\sqrt{14}}{3}\int_0^3\left(\frac{5}{2}(36 - 24y + 4y^2) + 12y - 4y^2 + 36 - 12y\right)dy =$$

$$= \frac{\sqrt{14}}{3}\int_0^3(6y^2 - 60y + 126)dy = 2\sqrt{14}\int_0^3(y^2 - 10y + 21)dy =$$

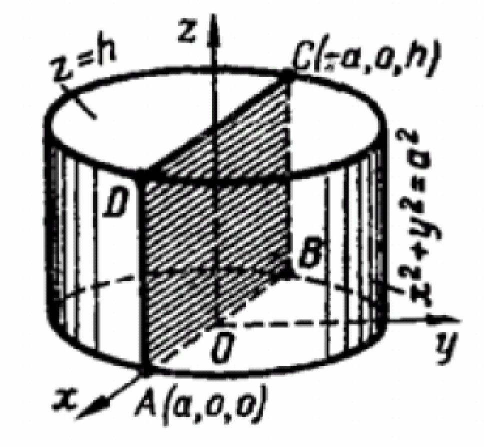
$$= 2\sqrt{14}\cdot\left(\frac{y^3}{3}\Big|_0^3 - 10\frac{y^2}{2}\Big|_0^3 + 21y\Big|_0^3\right) = 2\sqrt{14}\cdot(9 - 45 + 63) = 54\sqrt{14}$$

---

b) $K = \iint_{W}(y + z + \sqrt{a^2 - x^2})ds$ $x^2 + y^2 = a^2$ z = 0 i z = h

Skicirajmo oblast W (vidi sliku na sljedećoj stranici)

$$\iint_{W} f(x,y,z)ds = \iint_{D} f(x, y(x,z), z) \cdot \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dxdy$$



$$x^2+y^2=a^2$$
$$y^2=a^2-x^2$$

$$|y|=\sqrt{a^2-x^2}$$
$$y=\sqrt{a^2-x^2}$$
i
$$y=-\sqrt{a^2-x^2}$$

$$K=K_1+K_2$$
$$\frac{\partial y}{\partial x}=-\frac{x}{\sqrt{a^2-x^2}}$$
$$\frac{\partial y}{\partial z}=0$$

$$D:\begin{cases}-a\le x\le a\\ 0\le z\le h\end{cases}$$

$$ds=\sqrt{1+\left(-\frac{x}{\sqrt{a^2-x^2}}\right)^2}\,dxdz=\sqrt{1+\frac{x^2}{a^2-x^2}}\,dxdz=\frac{adxdz}{\sqrt{a^2-x^2}}$$

$$K_1=\iint_W\left(y+z+\sqrt{a^2-x^2}\right)ds=\iint_D\left(\sqrt{a^2-x^2}+z+\sqrt{a^2-x^2}\right)\frac{adxdz}{\sqrt{a^2-x^2}}=$$

$$=a\iint_D\left(2+\frac{z}{\sqrt{a^2-x^2}}\right)dxdz=2a\int_{-a}^{a}dx\int_0^h dz+a\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}\int_0^h zdz=$$

$$=2a\cdot 2a\cdot h+\frac{ah^2}{2}\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}=\left|\begin{array}{l}x=a\sin t\\ dx=a\cos tdt\\ x=a\Rightarrow t=\frac{\pi}{2}\\ y=-a\Rightarrow t=-\frac{\pi}{2}\end{array}\right|=4a^2h+\frac{ah^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{a\cos tdt}{\sqrt{a^2-a^2\sin^2 t}}=$$

$$=4a^2h+\frac{ah^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{a\cos tdt}{a\sqrt{1-\sin^2 t}}=4a^2h+\frac{ah^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos tdt}{\cos t}=4a^2h+\frac{ah^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}dt=$$

$$=4a^2h+\frac{a^2h}{2}\cdot t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=4a^2h+\frac{ah^2\pi}{2}$$

$$y=-\sqrt{a^2-x^2}$$
$$\frac{\partial y}{\partial x}=\frac{x}{\sqrt{a^2-x^2}}$$
$$\frac{\partial y}{\partial z}=0$$
$$ds=\frac{adxdy}{\sqrt{a^2-x^2}}$$

$$K_2=\iint_W\left(y+z+\sqrt{a^2-x^2}\right)ds=\iint_D\left(-\sqrt{a^2-x^2}+z+\sqrt{a^2-x^2}\right)\frac{adxdz}{\sqrt{a^2-x^2}}=$$

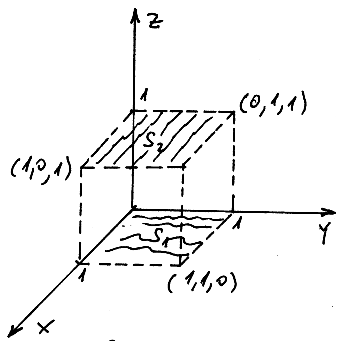
$$=\iint_D z\frac{adxdz}{\sqrt{a^2-x^2}}=a\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}\int_0^h zdz=a\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}\cdot\frac{z^2}{2}\Big|_0^h=\frac{ah^2}{2}\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}=$$

$$=\left|\begin{array}{l}x=a\sin t\\ dx=a\cos tdt\\ x=a\Rightarrow t=\frac{\pi}{2}\\ x=-a\Rightarrow t=-\frac{\pi}{2}\end{array}\right|=\frac{ah^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{a\cos tdt}{\sqrt{a^2-a^2\sin^2 t}}=\frac{ah^2}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}dt=\frac{ah^2}{2}\cdot t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{ah^2\pi}{2}$$

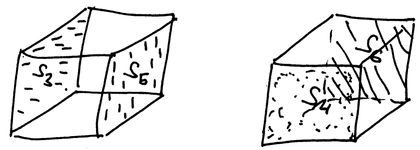
$$K=4a^2h+\frac{ah^2\pi}{2}+\frac{ah^2\pi}{2}=4a^2h+ah^2\pi=ah(4a+\pi h)$$

Izračunati površinski integral $\iint_S (x+y+z)\,dS$ gdje je S površina kocke $0\le x\le 1$, $0\le y\le 1$ i $0\le z\le 1$.

Rj.



Ako strane kocke označim sa S_1, S_2, S_3, S_4, S_5 i S_6



imamo:

$$\iint_S (x+y+z)\,dS = \iint_{S_1}(x+y+z)\,dx\,dy + \iint_{S_2}(x+y+z)\,dx\,dy + \iint_{S_3}(x+y+z)\,dx\,dz$$

$$+ \iint_{S_4}(x+y+z)\,dy\,dz + \iint_{S_5}(x+y+z)\,dx\,dz + \iint_{S_6}(x+y+z)\,dy\,dz$$

$$\iint_{S_1}(x+y+z)\,dx\,dy + \iint_{S_2}(x+y+z)\,dx\,dy = \int_0^1\Big[\int_0^1 (x+y)\,dy\Big]dx + \int_0^1\Big[\int_0^1 (x+y+1)\,dy\Big]dx =$$

$$\int_0^1 \Big(xy\Big|_0^1 + \frac12 y^2\Big|_0^1\Big)dx + \int_0^1\Big(xy\Big|_0^1 + \frac12 y^2\Big|_0^1 + y\Big|_0^1\Big)dx = \frac12 x^2\Big|_0^1 + \frac12 x\Big|_0^1 + \frac12 x\Big|_0^1$$

$$+ \frac12 x\Big|_0^1 + x\Big|_0^1 = 3$$

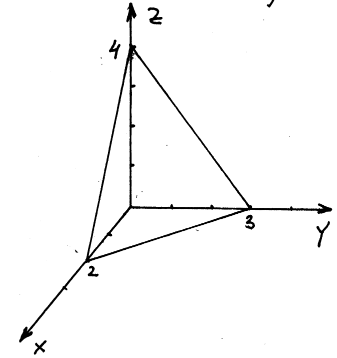
Prema tome: $\iint_S (x+y+z)\,dS = 9$.

Izračunati površinski integral $\iint_S \left(z+2x+\frac43 y\right)dS$ gdje je S dio ravni $\frac x2 + \frac y3 + \frac z4 = 1$ u prvom oktantu.

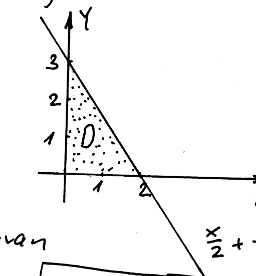
Rj. $\frac x2 + \frac y3 + \frac z4 = 1$ segmentni oblik jednačine ravni

$$\frac z4 = 1 - \frac x2 - \frac y3 \quad /\cdot 4$$

$$z = 4 - 2x - \frac43 y$$



Projekcija na xOy ravan izgleda



$$\frac{\partial z}{\partial x} = -2 \qquad \frac{\partial z}{\partial y} = -\frac43$$

$$\frac x2 + \frac y3 = 1$$

S' projekcija površi S na xOy ravan

$$I = \iint_S f(x,y,z)\,dS = \iint_{S'} f(x,y,z(x,y))\cdot\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+4+\frac{16}{9}} = \sqrt{\frac{61}{9}} = \frac{\sqrt{61}}{9}$$

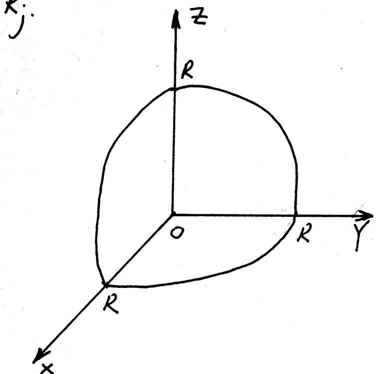
$$\iint_S \left(z+2x+\frac43 y\right)dS = \iint_D \left(4-2x-\frac43 y+2x+\frac43 y\right)\cdot\frac{\sqrt{61}}{3}\,dx\,dy = \frac{4\sqrt{61}}{3}\iint_D dx\,dy$$

(površina oblasti D)

$$= \frac{4\sqrt{61}}{3}\cdot\frac{2\cdot 3}{2} = 4\sqrt{61}.$$

Izračunati integral $I=\iint_S x\,dS$ gdje je S dio sfere $x^2+y^2+z^2=R^2$ u prvom oktantu.

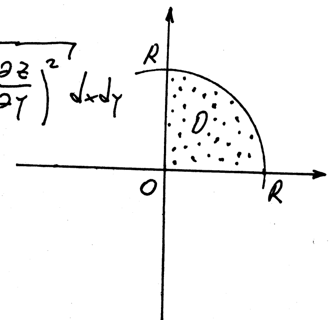
Rj.



$$x^2+y^2+z^2=R^2$$
$$z^2=R^2-x^2-y^2$$
$$z=\pm\sqrt{R^2-x^2-y^2}$$
$$z\ge 0 \qquad z=\sqrt{R^2-x^2-y^2}$$

Projekcija površi na xOy ravan

$$\iint_S f(x,y,z)\,dS=\iint_{S'} f(x,y,z(x,y))\cdot\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$



S' projekcija površi S na xOy ravan

$$\frac{\partial z}{\partial x}=\frac{-2x}{2\sqrt{R^2-x^2-y^2}}=\frac{-x}{\sqrt{R^2-x^2-y^2}}$$

$$\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{R^2-x^2-y^2}}$$

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+\frac{x^2}{R^2-x^2-y^2}+\frac{y^2}{R^2-x^2-y^2}}=\frac{R}{\sqrt{R^2-x^2-y^2}}$$

$$I=\iint_S x\,dS=\iint_D x\cdot\frac{R}{\sqrt{R^2-x^2-y^2}}\,dx\,dy$$

uvedimo polarne koordinate $x=r\cos\varphi$, $y=r\sin\varphi$ u našem slučaju

$$D':\begin{cases}x=r\cos\varphi\\ y=r\sin\varphi\\ 0\le\varphi\le\frac{\pi}{2}\\ 0\le r\le R\end{cases}\qquad dx\,dy=r\,dr\,d\varphi \qquad x^2+y^2=r^2$$

$$\iint_D \frac{xR}{\sqrt{R^2-x^2-y^2}}\,dx\,dy=\iint_{D'}\frac{r\cos\varphi R\cdot r}{\sqrt{R^2-r^2}}\,dr\,d\varphi$$

$$=R\int_0^{\pi/2}\cos\varphi\left[\int_0^R\frac{r^2}{\sqrt{R^2-r^2}}\,dr\right]d\varphi=\left|\begin{matrix}r=R\sin t\\ r=0\Rightarrow t=0\\ r=R\Rightarrow t=\frac{\pi}{2}\\ dr=R\cos t\,dt\end{matrix}\right|=R\int_0^{\pi/2}\cos\varphi\left[\int_0^{\pi/2}\frac{1}{R}\frac{R^2\sin^2 t}{\sqrt{1-\sin^2 t}}R\cos t\,dt\right]d\varphi$$

$$=R^3\int_0^{\pi/2}\cos\varphi\left[\int_0^{\pi/2}\sin^2 t\,dt\right]d\varphi=R^3\int_0^{\pi/2}\cos\varphi\left[\frac{1}{2}\int_0^{\pi/2}(1-\cos 2t)\,dt\right]=R^3\sin\varphi\Big|_0^{\pi/2}\cdot\frac{1}{2}\left(t\Big|_0^{\pi/2}-\frac{1}{2}\sin 2t\Big|_0^{\pi/2}\right)=\frac{R^3\pi}{4}$$

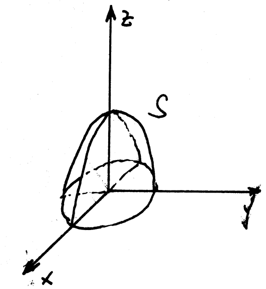
Izračunati površinski integral $\iint_S 3z\,dS$ gdje je S površina paraboloida $z=2-(x^2+y^2)$ iznad xy-ravni.

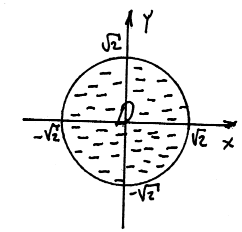
Rj. Neka je D projekcija površi S na xOy ravan. Tada

$$\iint_S f(x,y,z)\,dS=\iint_D f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

Pronađimo projekciju paraboloida $z=2-(x^2+y^2)$ na xOy ravan.

$z=0 \Rightarrow x^2+y^2=2$ krug sa centrom u tački $(0,0)$ poluprečnika $\sqrt{2}$





$$\frac{\partial z}{\partial x}=-2x \qquad \frac{\partial z}{\partial y}=-2y$$

$$I=\iint_S 3z\,dS=3\iint_D\left[2-(x^2+y^2)\right]\sqrt{1+4x^2+4y^2}\,dx\,dy$$

Da bi smo riješili ovaj dvostruki integral potrebno je uvesti smjenu promjenjivih.

Uvedimo polarne koordinate $x=r\cos\varphi$, $y=r\sin\varphi$

$$D':\begin{cases}0\le\varphi\le 2\pi\\ 0\le r\le\sqrt{2}\\ dx\,dy=r\,dr\,d\varphi\end{cases}$$ ove granice čitamo sa slike

$$x^2+y^2=r^2\cos^2\varphi+r^2\sin^2\varphi=r^2$$
$$1+4x^2+4y^2=1+4(x^2+y^2)=1+4r^2$$

$$I=3\iint_{D'}(2-r^2)\sqrt{1+4r^2}\cdot r\,dr\,d\varphi=3\iint_{D'}2r\sqrt{1+4r^2}\,dr\,d\varphi-3\iint_{D'}r^3\sqrt{1+4r^2}\,dr\,d\varphi$$

$$6\iint_{D'}r\sqrt{1+4r^2}\,dr\,d\varphi=6\int_0^{2\pi}\left[\int_0^{\sqrt{2}}r\sqrt{1+4r^2}\,dr\right]d\varphi=\left|\begin{matrix}1+4r^2=t^2 & r=0\Rightarrow t=1\\ 8r\,dr=2t\,dt & r=\sqrt{2}\Rightarrow t=3\\ r\,dr=\frac{1}{4}t\,dt & \end{matrix}\right|=$$

$$=6\int_0^{2\pi}\left[\int_1^3\frac{1}{4}t^2\,dt\right]d\varphi=6\cdot\frac{1}{4}\,\varphi\Big|_0^{2\pi}\cdot\frac{t^3}{3}\Big|_1^3=\frac{3}{2}\cdot\frac{1}{3}\cdot 2\pi\cdot 26=26\pi$$

$$3\iint_{D'}r^3\sqrt{1+4r^2}\,dr\,d\varphi=3\int_0^{2\pi}\left[\int_0^{\sqrt{2}}\underbrace{r^3}_{r^2\cdot r}\sqrt{1+4r^2}\,dr\right]d\varphi=\left|\begin{matrix}1+4r^2=t^2 & r\,dr=\frac{1}{4}t\,dt\\ 4r^2=t^2-1 & r=0\Rightarrow t=1\\ r^2=\frac{1}{4}(t^2-1) & r=\sqrt{2}\Rightarrow t=3\\ 8r\,dr=2t\,dt & \end{matrix}\right|=\ldots=\frac{111\pi}{10}$$

Zadaci za vježbu

U zadacima 3876—3884 izračunati date integrale.

3876. $\iint\limits_S \left(z+2x+\frac{4}{3}y\right) dq$, pri čemu je S deo ravni $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$ koji se nalazi u prvom oktantu.

3877. $\iint\limits_S xyz\,dq$, pri čemu je S deo ravni $x+y+z=1$ koja leži u prvom oktantu.

3878. $\iint\limits_S x\,dq$, pri čemu je S deo sfere $x^2+y^2+z^2=R^2$ koji se nalazi u prvom oktantu.

3879. $\iint\limits_S y\,dq$, po polusferi $z=\sqrt{R^2-x^2-y^2}$.

3880. $\iint\limits_S \sqrt{R^2-x^2-y^2}\,dq$ po polusferi $z=\sqrt{R^2-x^2-y^2}$.

3881. $\iint\limits_S x^2y^2\,dq$ po polusferi $z=\sqrt{R^2-x^2-y^2}$.

3882. $\iint\limits_S \frac{dq}{r^2}$, pri čemu je S deo cilindra $x^2+y^2=R^2$ ograničen ravnima $z=0$ i $z=H$, a r je odstojanje tačke na površini od koordinatnog početka.

3883. $\iint\limits_S \frac{dq}{r^n}$ po sferi $x^2+y^2+z^2=R^2$, pri čemu je r odstojanje tačke na sferi od nepomične tačke $P(0,\ 0,\ c)$, $(c>R)$.

3884. $\iint\limits_S \frac{dq}{r}$, pri čemu je S deo hiperboličnog paraboloida $z=xy$, isečen cilindrom $x^2+y^2=R^2$, a r je odstojanje tačke na površi S od z-ose.

3885*. Naći masu sfere ako je površinska gustina u svakoj njenoj tački brojno jednaka odstojanju te tačke od nekog određenog prečnika sfere.

3886. Naći masu sfere ako je površinska gustina u svakoj njenoj tački brojno jednaka kvadratu odstojanja te tačke od nekog određenog prečnika sfere.

Rješenja

3876. $4\sqrt{61}$. **3877.** $\frac{\sqrt{3}}{120}$. **3878.** $\frac{\pi R^3}{4}$.

3979. 0. **3880.** πR^3. **3881.** $\frac{2\pi R^6}{15}$.

3882. $2\pi \operatorname{arctg}\frac{H}{R}$. **3883.** $\frac{2\pi R}{c(n-2)}\left[\frac{1}{(c-R)^{n-2}}-\frac{1}{(c+R)^{n-2}}\right]$ za $n\neq 2$; $\frac{2\pi R}{c}\ln\frac{c+R}{c-R}$ za $n=2$.

3884. $\pi\left[R\sqrt{R^2+1}+\ln\left(R+\sqrt{R^2+1}\right)\right]$.

3885*. π^2R^3. Primeniti sferne koordinate. **3886.** $\frac{8}{3}\pi R^4$.

Površinski integrali II vrste

Obično su oblika: $\iint\limits_S P(x,y,z)\,dy\,dz + Q(x,y,z)\,dz\,dx + R(x,y,z)\,dx\,dy$

Uvijek ga svodimo na dvostruki integral.

S je neka data površina. Početni integral se obično podjeli na tri dijela $\iint\limits_S P(x,y,z)\,dy\,dz$, $\iint\limits_S Q(x,y,z)\,dz\,dx$ i $\iint\limits_S R(x,y,z)\,dx\,dy$.

$\vec{n}=(\cos\alpha, \cos\beta, \cos\gamma)$ je vektor normale na površinu S gdje su α, β i γ uglovi koje zaklapa vektor normale sa x, y i z osom.

Kad računamo $\iint\limits_S P(x,y,z)\,dy\,dz$ treba uzeti u obzir predznak broja $\cos\alpha$. Ako je $\cos\alpha<0$ ispred integrala stavljamo $-$ (minus), ako je $\cos\alpha>0$ ispred integrala stavljamo $+$ (plus) i ako je $\cos\alpha=0$ tada $\iint\limits_S P(x,y,z)\,dy\,dz=0$.

Analogno uzimamo vrijednost $\cos\beta$ za $\iint\limits_S Q(x,y,z)\,dz\,dx$ i $\cos\gamma$ za $\iint\limits_S R(x,y,z)\,dx\,dy$. $I=I_1+I_2+I_3$

Integral I_1 rješavamo projekcijom površi S na yOz ravan, integral I_2 projekcijom na xOz ravan i integral I_3 $I_3=\iint\limits_S R(x,y,z)\,dx\,dy$ projekcijom površi S na xOz ravan.

Kod površinskih integrala II vrste mora se označiti koju stranu površi uzimamo. Zavisi od toga sa koje strane vektor normale djeluje (ili sa unutrašnje ili sa spoljašnje oblasti površi).

Kod izbora površi S po kojoj se integrira mora se precizirati da li se uzima vanjska ili unutrašnja strana površi, jer prelaskom na suprotnu stranu integral mijenja predznak.

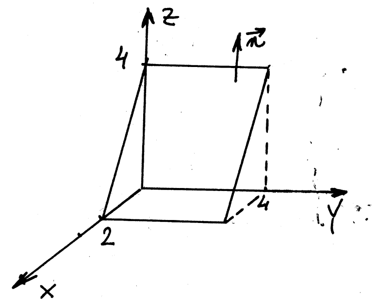
Izračunati $\iint_S z\,dx\,dy + x\,dz\,dx + y\,dy\,dz$ pri čemu je S gornja strana ravni $2x+z=4$, $0<y<4$ u prvom oktantu.

Rj.

$2x+z=4 \quad /:4$

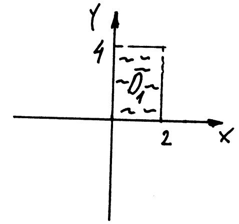
$\frac{x}{2}+\frac{z}{4}=1$ segmentni oblik jednačine ravni

$z=4-2x$



$\vec{n}=(2,0,1)$ vektor normale ravni

$|\vec{n}|=\sqrt{5}$ $\quad \vec{n}_0=\frac{\vec{n}}{|\vec{n}|}=\left(\frac{2}{\sqrt5},0,\frac{1}{\sqrt5}\right)$ $\quad$ $\cos\alpha \ \cos\beta \ \cos\gamma$



$I_1=\iint_S z\,dx\,dy$ projeciramo površ na xOy ravan $\qquad D_1:\begin{cases}0\le x\le 2\\ 0\le y\le 4\end{cases}$

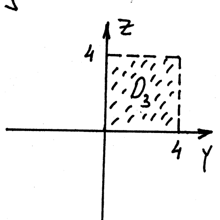
kako je $\cos\gamma>0 \Rightarrow I_1=+\iint_{D_1}(4-2x)\,dx\,dy=$

$=\int_0^4\left[\int_0^2(4-2x)\,dx\right]dy=\int_0^4\left(4x\Big|_0^2-2\cdot\frac12 x^2\Big|_0^2\right)dy=\int_0^4(8-4)\,dy=4y\Big|_0^4=16$

$I_2=\iint_S x\,dz\,dx$ (gledamo ugao β)

Kako je $\cos\beta=0 \Rightarrow I_2=0$

$I_3=\iint_S y\,dy\,dz$ (gledamo ugao α) $\quad \cos\alpha>0 \Rightarrow I_3=+\iint_{D_3} y\,dy\,dz$



$D_3:\begin{cases}0\le y\le 4\\ 0\le z\le 4\end{cases}$ $\qquad I_3=\int_0^4\left[\int_0^4 y\,dy\right]dz=\int_0^4 \frac12 y^2\Big|_0^4\,dz$

$=\frac12\cdot 16\cdot z\Big|_0^4=32$

$\iint_S z\,dx\,dy+x\,dz\,dx+y\,dy\,dz=16+0+32=48$

Izračunati površinski integral druge vrste

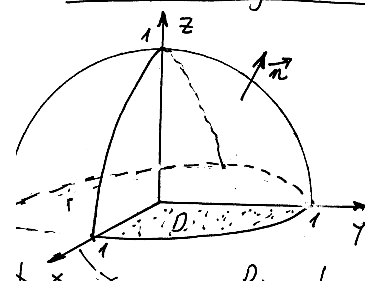
$$I=\iint_S xyz\,dx\,dy$$

gdje je S spoljna strana dijela sfere $x^2+y^2+z^2=1$, $x\ge 0$, $y\ge 0$.

Rj. Prisjetimo se: Neka je površ S data u obliku $z=\eta(x,y)$. Tada

$\iint_S R(x,y,z)\,dx\,dy=\pm\iint_D R(x,y,\eta(x,y))\,dx\,dy$ gdje

- $\pm$ zavisi od ugla koji vektor normale zaklapa sa z-osom, npr. $\vec{n}_0=(\cos\alpha,\cos\beta,\cos\gamma)$, $\quad \cos\gamma>0 \Rightarrow +$, $\quad \cos\gamma<0 \Rightarrow -$, $\quad \cos\gamma=0 \Rightarrow 0$
- D je ortogonalna projekcija površi S na xOy ravan



$z^2=1-x^2-y^2$

$z=\pm\sqrt{1-x^2-y^2}$

Kako je $x\ge0$, $y\ge0$ to je $z=\sqrt{1-x^2-y^2}$

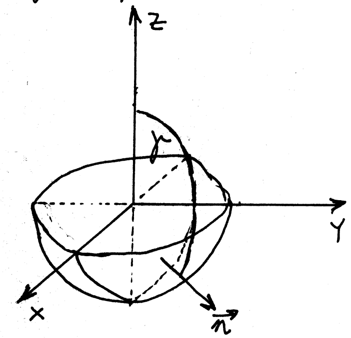
Primjetimo da je $0<\gamma<90^\circ \Rightarrow \cos\gamma>0$

$\iint_S xyz\,dx\,dy=\iint_D xy\sqrt{1-x^2-y^2}\,dx\,dy=$ $\left|\begin{array}{l}\text{uvedimo polarne koordinate}\\ x=r\cos\varphi \qquad 1-x^2-y^2=1-r^2\\ y=r\sin\varphi\\ dx\,dy=r\,dr\,d\varphi \qquad D\xrightarrow{\text{transf.}}D':\begin{cases}0\le r\le 1\\ 0\le\varphi\le\frac{\pi}{2}\end{cases}\end{array}\right|$

$=\iint_{D'} r^3\sin\varphi\cos\varphi\sqrt{1-r^2}\,dr\,d\varphi=\int_0^1 r^3\sqrt{1-r^2}\,dr\int_0^{\pi/2}\sin\varphi\cos\varphi\,d\varphi\overset{\text{za vježbu}}{=}\dots=\frac{2}{15}\cdot\frac12=\frac{1}{15}.$

Izračunati $\iint_S x^2y^2z\,dxdy$ gdje je S-vanjska strana donje polovine sfere $x^2+y^2+z^2=R^2$.

Rj.



Kako imamo $dxdy$: zanima nas ugao γ (γ je ugao koji vektor normale $\vec{n}$ na površ zaklapa sa z-osom).

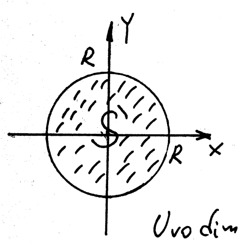
$$\gamma > \frac{\pi}{2} \Rightarrow \cos\gamma < 0$$

$$z^2=R^2-x^2-y^2$$

$$z<0 \qquad z=-\sqrt{R^2-x^2-y^2}$$

Da smo imali čitavu sferu tada bi integral podjelili na dva dijela za gornji i za donji dio sfere.

Gledamo projekciju površi S na xOy ravan:



$$S': x^2+y^2\le R^2$$

$$\iint_S x^2y^2z\,dxdy = -\iint_{S'} x^2y^2\left(-\sqrt{R^2-x^2-y^2}\right)dxdy$$

Uvodimo polarne koordinate
$x=r\cos\varphi$
$y=r\sin\varphi$

$$D:\begin{cases}0\le\varphi\le 2\pi\\ 0\le r\le R\\ dxdy=r\,d\varphi\,dr\end{cases} \qquad x^2+y^2=r^2$$

$$\iint_S x^2y^2z\,dxdy = \iint_{S'} x^2y^2\sqrt{R-x^2-y^2}\,dxdy = \iint_D r^2\cos^2\varphi\, r^2\sin^2\varphi\sqrt{R^2-r^2}\cdot r\,d\varphi\,dr$$

$$=\int_0^{2\pi}\cos^2\varphi\sin^2\varphi\left[\int_0^R r^5\sqrt{R^2-r^2}\,dr\right]d\varphi \overset{(*)}{=} \frac{8R^7}{105}\cdot\frac{\pi}{4}=\frac{2\pi R^7}{105}$$

(*)

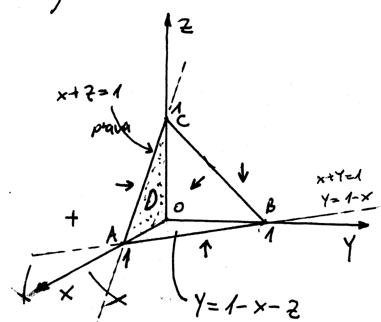
$$\int_0^R r^5\sqrt{R^2-r^2}\,dr = \int_0^R r^4\sqrt{R^2-r^2}\,r\,dr = \left|\begin{matrix}R^2-r^2=t^2 & r=0\Rightarrow t=R\\ -2r\,dr=2t\,dt & r=R\Rightarrow t=0\\ r\,dr=-t\,dt & \end{matrix}\right| = \int_0^R (R^2-t^2)^2\cdot\sqrt{t^2}\cdot t\,dt = \ldots = \frac{8R^7}{105}$$

$$\int_0^{2\pi}\cos^2\varphi\sin^2\varphi\,d\varphi = \int_0^{2\pi}\frac14(2\cos\varphi\sin\varphi)^2d\varphi = \frac14\int_0^{2\pi}\sin^2 2\varphi\,d\varphi = \frac14\int_0^{2\pi}\frac12(1-\cos4\varphi)\,d\varphi = \frac18\left(\varphi\Big|_0^{2\pi}-\frac14\sin4\varphi\Big|_0^{2\pi}\right)=\frac{\pi}{4}$$

Izračunati površinski integral $K=\oiint_{-W} y\,dxdz$ gdje je W-površina tetraedra ograničenog ravnima $x+y+z=1$, $x=0$, $y=0$ i $z=0$.

Rj: integral oblika $\iint_W R(x,y,z)\,dxdz$ zovemo površinski integral drugog tipa. Računamo ga tako što napravimo projekciju D površi W na xOz ravan i odredimo predznak broja $\cos\beta$ gdje je β ugao koji zaklapa vektor normale $\vec{n}$ površi W sa y-osom.

Skicirajmo naš tetraedar



Kako je u zadatku data oblast $-W$ to posmatramo vektore normale koje odgovaraju unutrašnjim površinama tetraedra

$$K=\oiint_{-W} y\,dxdz = \iint_{-\Delta AOC} y\,dxdz + \iint_{-\Delta AOB} y\,dxdz + \iint_{-\Delta BOC} y\,dxdz + \iint_{-\Delta ABC} y\,dxdz$$

$$\iint_{-\Delta AOC} y\,dxdz = +\iint_D 0\,dxdz = 0$$

$$\iint_{-\Delta AOB} y\,dxdz = \left|\begin{matrix}\text{vektor normale } \Delta AOB\\ \text{je okomit na y-osu}\end{matrix}\right| = 0$$

$$\iint_{-\Delta OBC} y\,dxdz = \left|\begin{matrix}\text{vektor normale } \Delta OBC\\ \text{je okomit na y-osu}\end{matrix}\right| = 0$$

$$\iint\limits_{-\Delta ABC} y\,dx\,dz = \left| \begin{array}{l} \text{vektor normale } \vec{n} \text{ na} \\ \Delta ABC \text{ sa Y-osom zaklapa} \\ \text{ugao } \beta \text{ koji je između } 90^\circ \text{ i } 180^\circ \\ \text{ZAŠTO? (vidi sliku)} \\ \cos\beta < 0 \end{array} \right| = -\iint\limits_D (1-x-z)\,dx\,dz =$$

$$= -\int_0^1 dx \int_0^{1-x} (1-x-z)\,dz = -\int_0^1 \left(z\Big|_0^{1-x} - xz\Big|_0^{1-x} - \frac{1}{2}z^2\Big|_0^{1-x} \right) dx =$$

$$= -\int_0^1 \left(1-x - x(1-x) - \frac{1}{2}\underbrace{(1-x)^2}_{1-2x+x^2} \right) dx = -\int_0^1 \left(1-x-x+x^2-\frac{1}{2}+x-\frac{1}{2}x^2 \right) dx$$

$$= -\int_0^1 \left(\frac{1}{2}x^2 - x + \frac{1}{2} \right) dx = -\left(\frac{1}{2}\cdot\frac{1}{3}x^3\Big|_0^1 - \frac{1}{2}x^2\Big|_0^1 + \frac{1}{2}x\Big|_0^1 \right) = -\left(\frac{1}{6} - \frac{1}{2} + \frac{1}{2} \right) = -\frac{1}{6}$$

traženo rješenje

II način

Možemo upotrebiti formulu Gauss-Ostrogradski

$$\iint\limits_S P(x,y,z)\,dy\,dz + Q(x,y,z)\,dx\,dz + R(x,y,z)\,dx\,dy = \iiint\limits_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx\,dy\,dz$$

Ω - oblast koju ograničava površ S

U našem slučaju $P(x,y,z) = R(x,y,z) = 0$

$$Q(x,y,z) = y \Rightarrow \frac{\partial Q}{\partial y} = 1$$

$$K = \oiint\limits_{-K} y\,dx\,dz = -\iiint\limits_\Omega dx\,dy\,dz = -\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz = -\int_0^1 dx \int_0^{1-x} (1-x-y)\,dy$$

$$= \left| \begin{array}{l} \text{primjetimo da smo sličan} \\ \text{integral već imali u prethodnom} \\ \text{slučaju} \end{array} \right| = \ldots = -\frac{1}{6}$$

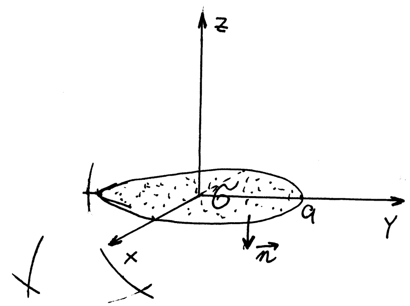
traženo rješenje

Izračunati površinski integral drugog tipa

(po koordinatama) $I = \iint\limits_\sigma \sqrt[4]{x^2+y^2}\,dx\,dy$ gdje je

σ - donja strana kruga $x^2+y^2 \le a^2$.

Rj: Skicirajmo datu površinu



> Prisjetimo se, kako se računa površinski integral drugog tipa, npr.
> $$\iint\limits_S R(x,y,z)\,dx\,dy$$
> posmatrano vektor normale $\vec{n}$ na površ S
> ako je $\cos\gamma < 0$ gdje je γ ugao između $\vec{n}$ i z-ose naš integral postaje
> $$\iint\limits_S R(x,y,z)\,dx\,dy = -\iint\limits_D R(x,y,z(x,y))\,dx\,dy$$
> gdje je D ortogonalna projekcija površi S a $z = z(x,y)$ jednačina površi S

U našem slučaju ortogonalna projekcija D je jednaka datoj površini σ.

Ugao γ je $\gamma = \pi$ (donja strana kruga) tj. $\cos\pi < 0$.

$$I = \iint\limits_\sigma \sqrt[4]{x^2+y^2}\,dx\,dy = -\iint\limits_D \sqrt[4]{x^2+y^2}\,dx\,dy = \left| \begin{array}{l} \text{uvodimo polarne koordinate} \\ x = r\cos\varphi \\ y = r\sin\varphi \quad D \xrightarrow{\text{transf.}} D': \begin{cases} 0 \le r \le a \\ 0 \le \varphi \le 2\pi \end{cases} \\ dx\,dy = r\,dr\,d\varphi \end{array} \right|$$

$$= -\iint\limits_{D'} \sqrt[4]{r^2}\,r\,dr\,d\varphi = -\int_0^{2\pi} d\varphi \int_0^a r^{\frac{3}{2}}\,dr = -\int_0^{2\pi} \frac{2}{5} r^{\frac{5}{2}}\Big|_0^a d\varphi = -\frac{2}{5} a^{\frac{5}{2}} \varphi\Big|_0^{2\pi}$$

$$I = -\frac{4}{5}\pi\sqrt{a^5}$$

traženo rješenje

(#) Izračunati površinski integral

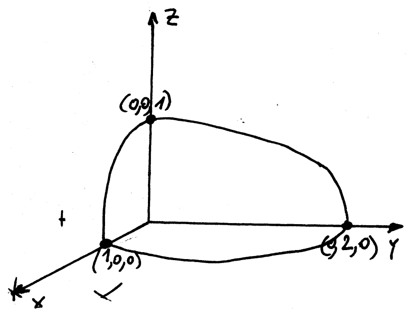
$$J=\iint_T 2\,dx\,dy + y\,dx\,dz - x^2 z\,dy\,dz$$

gdje je T vanjska strana elipsoida $4x^2+y^2+4z^2=4$ koji se nalazi u prvom oktantu.

Rj. skiciramo elipsoid

$4x^2+y^2+4z^2=4 \quad /:4$

$\frac{x^2}{1}+\frac{y^2}{4}+\frac{z^2}{1}=1$



$$J=\iint_T 2\,dx\,dy + y\,dx\,dz - x^2 z\,dy\,dz$$

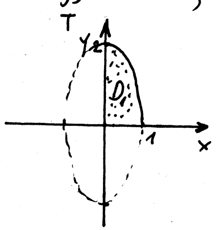
Ovo je površinski integral druge vrste. Prisjetimo se kako se računa npr. $\iint_T P(x,y,z)\,dy\,dz$. Neka je $\vec{n}$ vektor normale površi T koji sa x, y i z osom redom zaklapa uglove α, β i γ, i neka je D ortogonalna projekcija površi T na yOz ravan. Tada

$\iint_T P(x,y,z)\,dy\,dz = \pm\iint_D P(\varphi(y,z),y,z)\,dy\,dz$ gdje je + ako je $\cos\alpha>0$, − (minus) ako je $\cos\alpha<0$, a $x=\varphi(y,z)$ je jednačina koja opisuje površ T.

$$J=\iint_T 2\,dx\,dy + y\,dx\,dz - x^2 z\,dy\,dz = \iint_T 2\,dx\,dy + \iint_T y\,dx\,dz - \iint_T x^2 z\,dy\,dz = J_1+J_2-J_3$$

Izračunajmo redom J_1, J_2 i J_3.

$J_1=\iint_T 2\,dx\,dy$, vektor normale $\vec{n}$ na T sa z osom zaklapa ugao $\gamma\in(0,\frac{\pi}{2})$ tj. $\cos\gamma>0$



$z=0: \quad 4x^2+y^2=4$

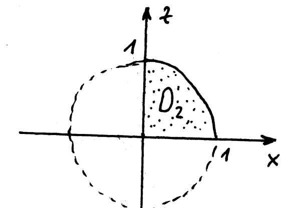
$D_1: \begin{cases} 0\le x\le 1 \\ 0\le y\le 2\sqrt{1-x^2}\end{cases}$

D_1 je četvrtina elipse

$P_{elipse}=ab\pi, \quad J_1=+2\iint_{D_1} dx\,dy = 2\cdot\frac{1}{4}P_{elipse}=\frac{1}{2}\cdot 2\pi=\pi$

$J_2=\iint_T y\,dx\,dz$, vektor normale $\vec{n}$ na površ T sa y-osom zaklapa uglove od 0 do $\pi/2$ (I oktant) pa je $\cos\gamma>0$.

Neka je D_2 ortogonalna projekcija površi T na xOz ravan.



$D_2: \quad 4x^2+4z^2=4$

$4x^2+y^2+4z^2=4$

$y^2=4-4x^2-4z^2$

$y=2\sqrt{1-x^2-z^2}$

$D_2: \begin{cases} 0\le x\le 1 \\ 0\le z\le\sqrt{1-x^2}\end{cases}$

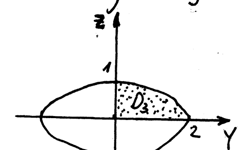
$$J_2=\iint_T y\,dx\,dz = +2\iint_{D_2}\sqrt{1-x^2-z^2}\,dx\,dz = \left|\begin{matrix}\text{uvedimo polarne koordinate} \\ x=r\cos\varphi \\ z=r\sin\varphi \\ dz\,dy=r\,dr\,d\varphi \\ D_2\to D_2'\end{matrix}\right| \quad D_2'=\begin{cases}0\le r\le 1\\ 0\le\varphi\le\frac{\pi}{2}\end{cases}$$

$$=2\iint_{D_2'}\sqrt{1-r^2\cos^2\varphi-r^2\sin^2\varphi}\;r\,dr\,d\varphi = 2\int_0^{\frac{\pi}{2}}d\varphi\int_0^1\sqrt{1-r^2}\;r\,dr=$$

$$=2\int_0^{\frac{\pi}{2}}d\varphi\int_0^1\sqrt{1-r^2}\left(-\frac{1}{2}\right)d(1-r^2) = -\varphi\Big|_0^{\frac{\pi}{2}}\cdot\frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^1 = -\frac{\pi}{2}\cdot\left(0-\frac{2}{3}\right)=\frac{\pi}{3}$$

$J_3=\iint_T x^2 z\,dy\,dz$, vektor normale $\vec{n}$ na površ T sa x-osom zaklapa uglove od 0 do $\frac{\pi}{2}$ pa je $\cos\alpha>0$

Neka je D_3 ortogonalna projekcija površi T na yOz ravan.



$D_3: \quad y^2+4z^2=4$

$\frac{y^2}{4}+\frac{z^2}{1}=1$

$y^2=4-4z^2$

$4x^2+y^2+4z^2=4$

$4x^2=4-y^2-4z^2$

$x^2=1-\frac{1}{4}y^2-z^2$

$$J_3=\iint_T x^2 z\,dy\,dz = +\iint_{D_3}\left(1-\frac{1}{4}y^2-z^2\right)z\,dy\,dz =$$

$$\left|D_3:\begin{cases}0\le z\le 1\\ 0\le y\le 2\sqrt{1-z^2}\end{cases}\right| = \int_0^1 z\,dz\int_0^{2\sqrt{1-z^2}}\left(1-\frac{1}{4}y^2-z^2\right)dy = \int_0^1 z\left(y\Big|_0^{2\sqrt{1-z^2}}-\frac{1}{4}\cdot\frac{1}{3}y^3\Big|_0^{2\sqrt{1-z^2}}-z^2y\Big|_0^{2\sqrt{1-z^2}}\right)$$

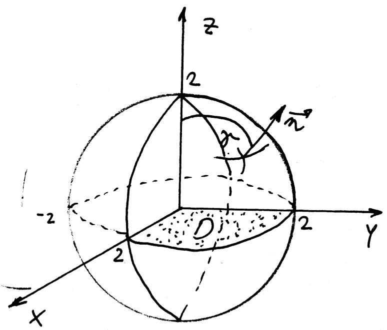
izvučemo $2\sqrt{1-z^2}$

$$=\int_0^1 z\left(2\sqrt{1-z^2}-\frac{2}{3}\sqrt{(1-z^2)^3}-2z^2\sqrt{1-z^2}\right)dz = \frac{4}{3}\int_0^1 z(1-z^2)^{\frac{3}{2}}dz = \frac{2}{3}\cdot\frac{2(1-z^2)^{\frac{5}{2}}}{5}\Big|_1^0=\frac{4}{15}$$

Prema tome $J=\pi+\frac{\pi}{3}-\frac{4}{15}=\frac{4\pi}{3}-\frac{4}{15}$.

Izračunati površinski integral $I=\iint\limits_S xy^3z\,dxdy$, ako je S vanjska strana sfere $x^2+y^2+z^2=4$ u prvom oktantu.

Rj. $x^2+y^2+z^2=4$ je sfera sa centrom u koordinatnom početku čiji je poluprečnik dužine 2.



Kad računamo $\iint\limits_S f(x,y,z)\,dxdy$ treba uzeti u obzir predznak broja $\cos\gamma$. Ako je $\cos\gamma<0$ ispred integrala stavljamo minus, ako je $\cos\gamma>0$ ispred integrala stavljamo plus, a ako je $\cos\gamma=0$ tada je integral jednak 0. γ je ugao koji vektor normale $\vec{n}$ ($\vec{n}=(\cos\alpha,\cos\beta,\cos\gamma)$) zaklapa sa z-osom

Vektor normale $\vec{n}$ je u prvom oktantu $\Rightarrow 0<\gamma<\frac{\pi}{2}$

$\Rightarrow \cos\gamma>0$

$$x^2+y^2+z^2=4$$

$$z=\pm\sqrt{4-(x^2+y^2)} \quad \text{nama treba } +$$

$$I=\iint\limits_S xy^3z\,dxdy=\iint\limits_D xy^3\left(\sqrt{4-(x^2+y^2)}\right)dxdy=\left|\begin{array}{l}\text{uvedimo polarne koordinate}\\ x=r\cos\varphi \\ y=r\sin\varphi \\ dxdy=r\,dr\,d\varphi \\ x^2+y^2=r^2\end{array} \quad D':\begin{cases}0\le\varphi\le\frac{\pi}{2}\\ 0\le r\le 2\end{cases}\right|$$

$$=\iint\limits_{D'} r\cos\varphi\, r^3\sin^3\varphi\sqrt{4-r^2}\,r\,dr\,d\varphi=\int_0^{\frac{\pi}{2}}\cos\varphi\sin^3\varphi\,d\varphi\int_0^2 r^5\sqrt{4-r^2}\,dr=I_1\cdot I_2$$

$$I_1=\int_0^{\frac{\pi}{2}}\cos\varphi\cdot\sin^3\varphi\,d\varphi=\left|\begin{array}{l}\sin\varphi=t\\ \cos\varphi d\varphi=dt\\ \varphi\big|_0^{\frac{\pi}{2}}\Rightarrow t\big|_0^1\end{array}\right|=\int_0^1 t^3dt=\frac{1}{4}t^4\Big|_0^1=\frac{1}{4}$$

$$I_2=\int_0^2 r^5\sqrt{4-r^2}\,dr=\int_0^2 r^4\sqrt{4-r^2}\,r\,dr=\left|\begin{array}{l}4-r^2=t^2\\ -2r\,dr=2t\,dt\\ r\,dr=-t\,dt\end{array}\quad r\big|_0^2\Rightarrow t\big|_2^0\right|=\int_0^2(4-t^2)^2\cdot t\,t\,dt$$

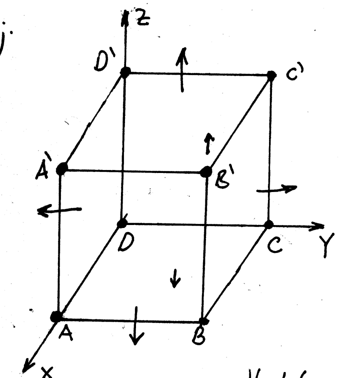
$$=\int_0^2(16-8t^2+t^4)\cdot t^2\,dt=\int_0^2(t^6-8t^4+16t^2)\,dt=\ldots=\frac{1024}{105}$$

$$I=\frac{1}{4}\cdot\frac{1024}{105}=\frac{256}{105}$$

traženo rješenje

Izračunati integral $\iint\limits_S x\,dydz+y\,dzdx+z\,dxdy$ gdje je S vanjska strana kocke koju čine ravni $x=0$, $y=0$, $z=0$, $x=1$, $y=1$, $z=1$.

Rj.



Označimo sa $I_1=\iint\limits_S x\,dydz$

Ovaj integral vadimo po šest površina: ABCD, ABB'A', BCC'B', ADD'A', A'B'C'D' i DCC'D'.

Kako imamo $dydz$ posmatramo ugao α koji zaklapa vektor normale na površ sa x osom

Vektor normala površina ABCD, A'B'C'D', BCC'B' i ADD'A' je okomita na x-osu $\Rightarrow$

$$\Rightarrow \iint\limits_{ABCD}x\,dydz=\iint\limits_{A'B'C'D'}x\,dydz=\iint\limits_{BCC'B'}x\,dydz=\iint\limits_{ADD'A'}x\,dydz=0$$

Kako je $x=0$ za površinu DCC'D' $\Rightarrow \iint\limits_{DCC'D'}x\,dydz=0$

Za I_1 ostaje nam samo površina ABB'A'

$\vec{n}_0=(1,0,0) \Rightarrow \cos\alpha>0 \Rightarrow I_1=+\iint\limits_D dydz$

gdje je D oblast dobijena projekcijom kvadrata ABB'A' na yOz ravan $D:\begin{cases}0\le y\le 1\\ 0\le z\le 1\end{cases}$

$$I_1=\iint\limits_D dydz=\int_0^1\left[\int_0^1 dy\right]dz=z\Big|_0^1\,y\Big|_0^1=1$$

Sad nije teško, analognim zaključivanjem, vidjeti da je $\iint\limits_S y\,dzdx=1$ i $\iint\limits_S z\,dxdy=1$ redom po površinama BCC'B' i A'B'C'D' dok je po ostalim površinama $=0$ $\Rightarrow \iint\limits_S x\,dydz+y\,dzdx+z\,dxdy=3$

(#) Izračunati površinski integral druge vrste

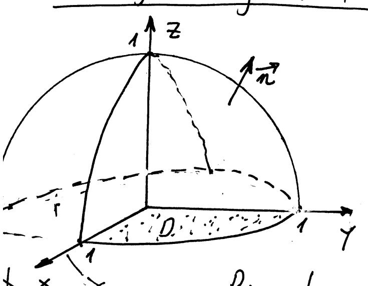
$$I=\iint_S xyz\,dx\,dy$$

gdje je S spoljna strana dijela sfere $x^2+y^2+z^2=1$, $x\geq 0$, $y\geq 0$.

Rj. Prisjetimo se: Neka je površ S data u obliku $z=\eta(x,y)$. Tada

$$\iint_S R(x,y,z)\,dx\,dy=\pm\iint_D R(x,y,\eta(x,y))\,dx\,dy$$ gdje

- $\pm$ zavisi od ugla koji vektor normale zaklapa sa z-osom, npr. $\vec{n}_0=(\cos\alpha,\cos\beta,\cos\gamma)$, $\cos\gamma>0 \Rightarrow +$, $\cos\gamma<0 \Rightarrow -$, $\cos\gamma=0 \Rightarrow 0$
- D je ortogonalna projekcija površi S na xOy ravan



$z^2=1-x^2-y^2$

$z=\pm\sqrt{1-x^2-y^2}$

kako je $x\geq 0$, $y\geq 0$ to je $z=\sqrt{1-x^2-y^2}$

Primjetimo da je $0<\gamma<90^\circ \Rightarrow \cos\gamma>0$

$$\iint_S xyz\,dx\,dy=\iint_D xy\sqrt{1-x^2-y^2}\,dx\,dy=$$

| uvedimo polarne koordinate: $x=r\cos\varphi$, $y=r\sin\varphi$, $dx\,dy=r\,dr\,d\varphi$, $1-x^2-y^2=1-r^2$, $D\overset{\text{transf.}}{\rightsquigarrow}D'$: $\begin{cases}0\leq r\leq 1\\ 0\leq\varphi\leq\frac{\pi}{2}\end{cases}$ |

$$=\iint_{D'} r^3\sin\varphi\cos\varphi\sqrt{1-r^2}\,dr\,d\varphi=\int_0^1 r^3\sqrt{1-r^2}\,dr\int_0^{\pi/2}\sin\varphi\cos\varphi\,d\varphi\overset{\text{za vježbu}}{=}\ldots=\frac{2}{15}\cdot\frac{1}{2}=\frac{1}{15}.$$

Zadaci za vježbu

U zadacima 3887—3893 izračunati date površinske integrale.

3887. $\iint_S x\,dy\,dz+y\,dx\,dz+z\,dx\,dy$ po spoljnoj strani kocke obrazovane ravnima $x=0$, $y=0$, $z=0$, $x=1$, $y=1$, $z=1$.

3888. $\iint_S x^2y^2z\,dx\,dy$ po spoljnoj strani donje polovine sfere $x^2+y^2+z^2=R^2$.

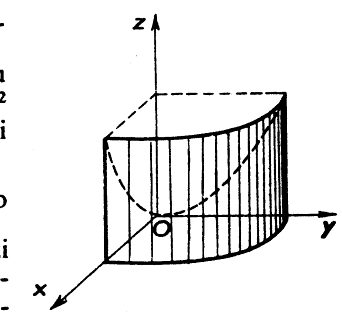
3889. $\iint_S z\,dx\,dy$ po spoljnoj strani elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

3890. $\iint_S z^2\,dx\,dy$ po spoljnoj strani elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

3891. $\iint_S xz\,dx\,dy+xy\,dy\,dz+yz\,dx\,dz$ po spoljnoj strani piramide obrazovane ravnima $x=0$, $y=0$, $z=0$ i $x+y+z=1$.

3892. $\iint_S yz\,dx\,dy+xz\,dy\,dz+xy\,dx\,dz$ po spoljnoj strani zatvorene površine koja se nalazi u prvom oktantu a sastoji se iz dela cilindra $x^2+y^2=R^2$ i odgovarajućih delova ravni $x=0$, $y=0$, $z=0$ i $z=H$.

3893. $\iint_S y^2z\,dx\,dy+xz\,dy\,dz+x^2y\,dx\,dz$ po spoljnoj strani zatvorene površine koja se nalazi u prvom oktantu a sastoji se iz obrtnog paraboloida $z=x^2+y^2$, cilindra $x^2+y^2=1$ i odgovarajućih delova koordinatnih ravni (sl. 68).



Sl. 68

Rješenja

3887. 3. **3888.** $\frac{2\pi R^7}{105}$. **3889.** $\frac{4}{3}\pi abc$. **3890.** 0. **3891.** $\frac{1}{8}$. **3892.** $R^2H\left(\frac{2R}{3}+\frac{\pi H}{8}\right)$. **3893.** $\frac{\pi}{8}$.

Primjena površinskih integrala

I Izračunavanje površine dijela glatke površi, koja pripada prostoru $\mathbb{R}^3$

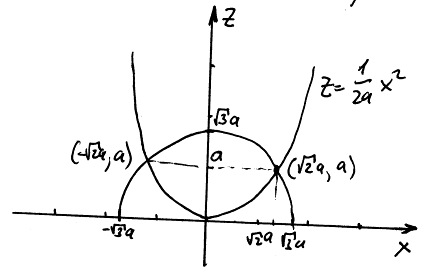
Neka je površ S zadana jednačinom $z = z(x,y)$ gdje su $(x,y) \in D$. (D - je oblast u ravni xOy u koju se projektuje površ $z = z(x,y)$).

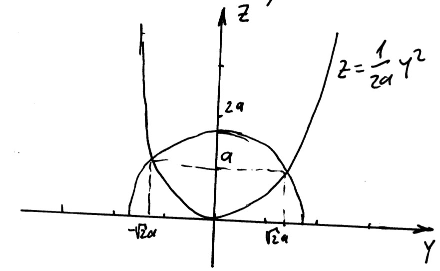
Površina P dijela glatke površi $S \subseteq \mathbb{R}^3$ računa se po formuli

$$P = \iint_S dS = \iint_D \sqrt{1 + (z'_x)^2 + (z'_y)^2}\, dx\, dy.$$

#) Izračunati površinu dijela lopte $x^2+y^2+z^2 = 3a^2$ koja se nalazi ispod parabole $x^2+y^2 = 2az$ a iznad xOy ravni.

Rj. Na osnovu skica presjeka datih površina sa xOz i yOz ravnima ćemo vidjeti kako tijela su u pitanju.





$$x^2 + z^2 = 3a^2$$
$$x^2 = 2az$$

$$z^2 + 2az - 3a^2 = 0$$
$$D = 4a^2 + 12a^2 = 16a^2$$
$$z_{1,2} = \frac{-2a \pm 4a}{2}$$
$$z_1 = a \quad z_2 = -3a$$

$P = \iint_S dS$ površinski integral prve vrste

$$z^2 = 3a^2 - x^2 - y^2$$
$$z = \pm\sqrt{3a^2 - x^2 - y^2}$$

U našem slučaju S je $z = \sqrt{3a^2 - x^2 - y^2}$ i to dio ove površine koji se nalazi ispod parabole

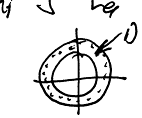
$$P = \iint_S dS = \iint_D \sqrt{1 + (z'_x)^2 + (z'_y)^2}\, dx\, dy$$

$$z'_x = \frac{-2x}{2\sqrt{3a^2 - x^2 - y^2}}, \quad z'_y = \frac{-y}{\sqrt{3a^2 - x^2 - y^2}}$$

$$1 + (z'_x)^2 + (z'_y)^2 = 1 + \frac{x^2}{3a^2 - x^2 - y^2} + \frac{y^2}{3a^2 - x^2 - y^2} = \frac{3a^2}{3a^2 - x^2 - y^2}$$

$$P = \sqrt{3}a \iint_D \frac{dx\, dy}{\sqrt{3a^2 - x^2 - y^2}}$$

gdje je D projekcija površi S na xOy ravan. U našem slučaju



Uvedimo polarne koordinate

$x = r\cos\varphi$

$y = r\sin\varphi$

$dx\,dy = r\,dr\,d\varphi$

$$D \xrightarrow{\text{transformacija}} D' : \begin{cases} \sqrt{2}a \le r \le \sqrt{3}a \\ 0 \le \varphi \le 2\pi \end{cases}$$

$x^2 + y^2 = r^2$

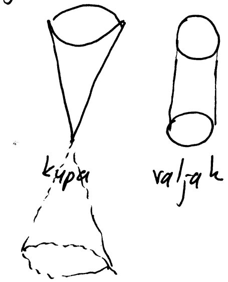
$$P = \sqrt{3}a \iint_{D'} \frac{r\,dr\,d\varphi}{\sqrt{3a^2 - r^2}} = \sqrt{3}a \int_0^{2\pi} d\varphi \int_{\sqrt{2}a}^{\sqrt{3}a} \frac{r\,dr}{\sqrt{3a^2-r^2}} = \left| \begin{array}{l} 3a^2 - r^2 = t^2 \\ -2r\,dr = 2t\,dt \\ r\Big|_{\sqrt{2}a}^{\sqrt{3}a} \Rightarrow t\Big|_a^0 \end{array} \right|$$

$$= \sqrt{3}a \int_0^{2\pi} d\varphi \int_0^a \frac{t\,dt}{t} = 2a^2\sqrt{3}\pi$$ traženo rješenje

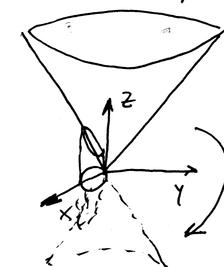
$2\pi \quad a$

(#) Izračunati površinu onog dijela kupe $z^2 = x^2 + y^2$ koji se nalazi unutar valjka $x^2 + y^2 = 2x$.

Rj.



Prema zadatku dio kupe se nalazi unutar valjka



$x^2 + y^2 = 2x$

$x^2 - 2\cdot x \cdot 1 + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$

sličnu figuru ćemo imati i sa druge strane xOy ravni

$$P = \iint_S dS$$ gdje je S dio kupe koji se nalazi unutar valjka

$$P = \iint_D \sqrt{1 + z_x'^2 + z_y'^2}\,dx\,dy$$

$z = \pm\sqrt{x^2+y^2}$

Ako za z uzmemo $z = \sqrt{x^2+y^2}$ dobićemo površinu dijela kupe iznad xOy ravni.

$z = \sqrt{x^2+y^2}$

$$z_x' = \frac{2x}{2\sqrt{x^2+y^2}}, \quad z_y' = \frac{y}{\sqrt{x^2+y^2}}, \quad 1 + z_x'^2 + z_y'^2 = 1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} = 1 + 1 = 2$$

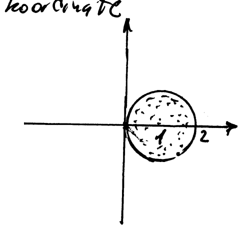
D: unutrašnjost kruga $x^2 + y^2 = 2x$

uvedimo polarne koordinate

$x = r\cos\varphi + 1$

$y = r\sin\varphi$

$dx\,dy = r\,dr\,d\varphi$



$$D \xrightarrow{\text{transf.}} D' : \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le 2\pi \end{cases}$$

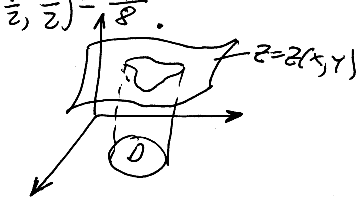
$$\frac{1}{2}P = \iint_D \sqrt{2}\,dx\,dy = \sqrt{2}\iint_{D'} r\,dr\,d\varphi = \sqrt{2}\int_0^{2\pi} d\varphi \int_0^1 r\,dr = \sqrt{2}\cdot \frac{1}{2}r^2\Big|_0^1 \varphi\Big|_0^{2\pi} = \sqrt{2}\pi$$

$P = 2\sqrt{2}\pi$

Izračunati površinu dijela površi $S: z^2=2xy$ određene u prvom oktantu u presjeka sa ravnima: $x=0,\ y=0$ i $x+y=1$.

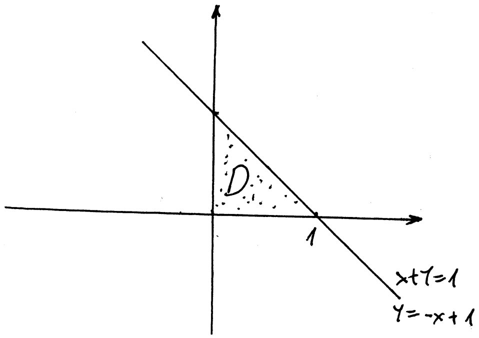
Rj. Uputa: $B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$, $B(\frac{3}{2},\frac{3}{2})=\frac{\pi}{8}$, $B(\frac{1}{2},\frac{5}{2})=\frac{3\pi}{8}$.

$$P=\iint_S dS=\iint_D \sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$



Kako je površina S u prvom oktantu, u našem slučaju je

$S:\ z=\sqrt{2}\sqrt{xy}$



$$z'_x=\sqrt{2}\,\frac{y}{2\sqrt{xy}}$$

$$z'_y=\sqrt{2}\,\frac{x}{2\sqrt{xy}}$$

$$1+(z'_x)^2+(z'_y)^2=1+\frac{y^2}{2xy}+\frac{x^2}{2xy}=\frac{2xy+y^2+x^2}{2xy}=\frac{(x+y)^2}{2xy}$$

$$P=\iint_S dS=\iint_D \frac{x+y}{\sqrt{2xy}}\,dx\,dy=\frac{1}{\sqrt{2}}\int_0^1 dx\int_0^{-x+1}\frac{(x+y)}{\sqrt{xy}}\,dy=$$

$$=\frac{1}{\sqrt{2}}\int_0^1 dx\int_0^{1-x}\left(x\cdot x^{-\frac{1}{2}}\cdot y^{-\frac{1}{2}}+y\cdot x^{-\frac{1}{2}}\cdot y^{-\frac{1}{2}}\right)dy=\frac{1}{\sqrt{2}}\int_0^1 dx\int_0^{1-x}\left(x^{\frac{1}{2}}y^{-\frac{1}{2}}+x^{-\frac{1}{2}}y^{\frac{1}{2}}\right)dy$$

$$=\frac{1}{\sqrt{2}}\int_0^1\left(x^{\frac{1}{2}}\,\frac{y^{\frac{1}{2}}}{\frac{1}{2}}\Big|_0^{1-x}+x^{-\frac{1}{2}}\cdot\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^{1-x}\right)dx=\frac{2}{\sqrt{2}}\int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}dx+$$

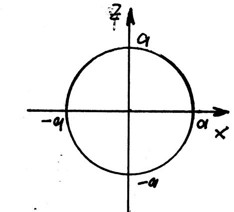
$$+\frac{2}{3\sqrt{2}}\int_0^1 x^{-\frac{1}{2}}(1-x)^{\frac{3}{2}}dx=\sqrt{2}\int_0^1 x^{\frac{3}{2}-1}(1-x)^{\frac{3}{2}-1}dx+\frac{\sqrt{2}}{3}\int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{5}{2}-1}dx=\sqrt{2}\,B\left(\frac{3}{2},\frac{3}{2}\right)+$$

$$+\frac{\sqrt{2}}{3}B\left(\frac{1}{2},\frac{5}{2}\right)=\frac{\pi}{2\sqrt{2}}.$$

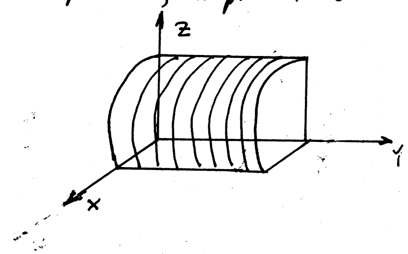
Neka je S površina tijela koje je dobijeno presjekom dva cilindra $S_1=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+z^2=a^2,\ y\in\mathbb{R}\}$ i $S_2=\{(x,y,z)\in\mathbb{R}^3 \mid y^2+z^2=a^2,\ x\in\mathbb{R}\}$. Izračunati površinu dobijenog tijela.

Rj. $P=\iint_S dS$ Skicirajmo S_1 i S_2, pa skicirajmo njihov presjek.

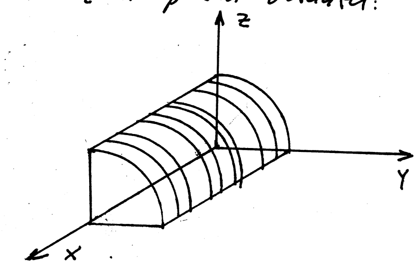
$S_1:\ x^2+z^2=a^2$ u ravni: xOz



U prostoru, u prvom oktantu:



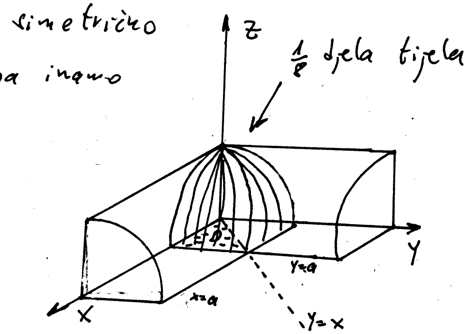
S_2 u prvom oktantu:



Presjek $S_1\cap S_2$ će kao rezultat dati tijelo koje je simetrično u odnosu na sve tri ravni xOy, xOz i yOz.

$\frac{1}{8}$ dijela tijela će se nalaziti u prvom oktantu:

Primjetimo da je i ovo tijelo simetrično u odnosu na pravu $y=x$ pa imamo



$$P=\frac{1}{16}\iint_D\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$$

gdje je $D:\begin{cases}0\le x\le a\\ 0\le y\le x\end{cases}$

$z^2=a^2-x^2$ tj. $z=\sqrt{a^2-x^2}$.

$z'_x=\frac{-x}{\sqrt{a^2-x^2}}$, $z'_y=0$ $\quad 1+(z'_x)^2+(z'_y)^2=1+\frac{x^2}{a^2-x^2}=\frac{a^2}{a^2-x^2}$

tražena površina

$$P=16a\int_0^a\frac{dx}{\sqrt{a^2-x^2}}\int_0^x dy=16a\int_0^a\frac{x\,dx}{\sqrt{a^2-x^2}}=\left|\begin{array}{l}a^2-x^2=t\\ -2x\,dx=dt\\ x\,dx=-\frac{1}{2}dt\\ \dots\end{array}\right|=\dots=16a\sqrt{a^2-x^2}\Big|_a^0=16a^2$$

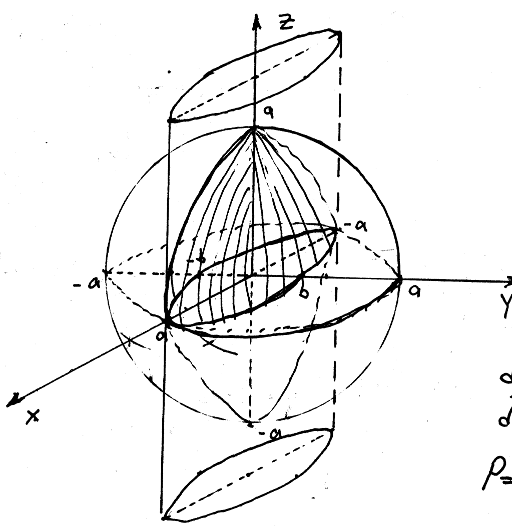
\# Izračunati površinu dijela sfere

$$S=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=a^2\}$$

koji se nalazi u unutrašnjosti cilindra

$$S_1=\{(x,y,z)\in\mathbb{R}^3 \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z\in\mathbb{R}\},\quad b\le a$$

R: Skiciramo sferu S i cilindar S_1.



Cilindrična površ u presjeku sa sferom, isjeca iz nje simetričnu površ u odnosu na ravan xOy. Ta dva simetrična dijela označimo sa l_1 i l_2. Svaku od ova dva dijela, koordinatne ravni xOz i yOz ih dijele na četiri jednaka dijela.

$P=\iint\limits_S dS$ gdje je S površina dijela sfere ograničena cilindrom.

$S:\ x^2+y^2+z^2=a^2$

$$z=\pm\sqrt{a^2-x^2-y^2}$$

Zbog navedene simetričnosti posmatraću sferu samo u prvom oktantu

$$z=\sqrt{a^2-x^2-y^2}\qquad z'_x=\frac{-x}{\sqrt{a^2-x^2-y^2}},\quad z'_y=\frac{-y}{\sqrt{a^2-x^2-y^2}}$$

$\iint\limits_S dS=\iint\limits_D\sqrt{1+(z'_x)^2+(z'_y)^2}\,dx\,dy$ gdje je D projekcija površi S na xOy ravan

$$1+(z'_x)^2+(z'_y)^2=1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}=\frac{a^2}{a^2-x^2-y^2}$$

$$P=8\iint\limits_{\substack{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\\ a\ge0,\ b\ge0}}\sqrt{\frac{a^2}{a^2-x^2-y^2}}\,dx\,dy=8a\iint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le1,\ a\ge0,\ b\ge0}\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}=8a\iint\limits_D\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$$

gdje je $D:\begin{cases}\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\\ x\ge0,\ y\ge0\end{cases}$ ili drugačije napisano $D:\begin{cases}0\le x\le a\\ 0\le y\le\frac{b}{a}\sqrt{a^2-x^2}\end{cases}$

$$\frac{y^2}{b^2}\le 1-\frac{x^2}{a^2}$$

$$y^2\le\frac{b^2}{a^2}(a^2-x^2)$$

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$P=8a\int_0^a dx\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\frac{dy}{\sqrt{\underbrace{a^2-x^2}_{\text{ovo je broj za } dy}-y^2}}=8a\int_0^a\left(\arcsin\frac{y}{\sqrt{a^2-x^2}}\Bigg|_{y=0}^{y=\frac{b}{a}\sqrt{a^2-x^2}}\right)dx$$

$$\int\frac{dx}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}+c$$

$$=8a\int_0^a\Big(\arcsin\frac{b}{a}-\underbrace{\arcsin 0}_{=0}\Big)dx=$$

$$=8a\arcsin\frac{b}{a}\int_0^a dx=8a^2\arcsin\frac{b}{a}$$ tražena površina

Zadaci za vježbu i rješenja

(#) Izračunati površinu površi S, ako je S dio površi $z=\dfrac{x+y}{x^2+y^2}$

između cilindara $x^2+y^2=1$ i $x^2+y^2=2$ u I oktantu.

Rj.

Za površ $z=\dfrac{x+y}{x^2+y^2}$ imamo

$$\frac{\partial z}{\partial x}=\frac{y^2-2xy-x^2}{(x^2+y^2)^2},\quad \frac{\partial z}{\partial y}=\frac{x^2-2xy-y^2}{(x^2+y^2)^2},$$

pa je

$$P=\iint_S dS=\iint_D\sqrt{1+\left[\frac{y^2-2xy-x^2}{(x^2+y^2)^2}\right]^2+\left[\frac{x^2-2xy-y^2}{(x^2+y^2)^2}\right]^2}\,dxdy,$$

gdje je $D: 1\leqslant x^2+y^2\leqslant 2$.
Uvedimo polarne koordinate: $x=\rho\cos\varphi$, $y=\rho\sin\varphi$. Imaćemo:

$$P=\iint_D\sqrt{1+\frac{1}{\rho^4}(-\cos 2\varphi-\sin 2\varphi)^2+\frac{1}{\rho^4}(\cos 2\varphi-\sin 2\varphi)^2}\,d\rho\,d\varphi=$$

$$=\int_0^{\pi/2}d\varphi\int_1^{\sqrt{2}}\frac{\sqrt{\rho^4+2}}{\rho^4}\cdot\rho^3\,d\rho=\frac{\pi}{2}\int_{\sqrt{3}}^{\sqrt{6}}\frac{t^2}{2(t^2-2)}\,dt=$$

$$\left(2+\rho^4=t^2,\ \rho^3\,d=\frac{1}{2}\,t\,dt\right)$$

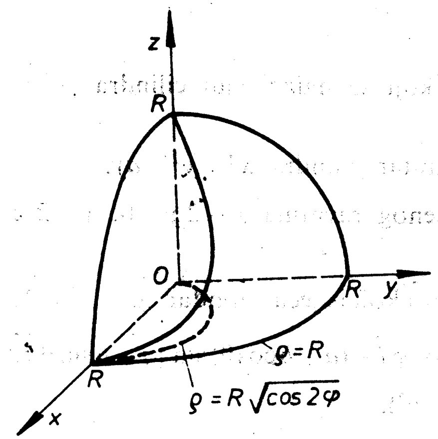
$$=\frac{\pi}{4}\int_{\sqrt{3}}^{\sqrt{6}}\left(1+\frac{2}{t^2-2}\right)dt=\frac{\pi}{4}\left(t+\frac{1}{\sqrt{2}}\ln\frac{t-\sqrt{2}}{t+\sqrt{2}}\right)\Bigg|_{\sqrt{3}}^{\sqrt{6}}=$$

$$=\frac{\pi}{4}\left[\sqrt{6}-\sqrt{3}-\frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\right)\right].$$

(#) Izračunati površinu površi S, ako je S dio površi $x^2+y^2+z^2=R^2$ **koji se nalazi van cilindra** $(x^2+y^2)^2=R^2(x^2-y^2)$.

Rj.

Površ $S: x^2+y^2+z^2=R^2$ isječena cilindrom $(x^2+y^2)^2=R^2(x^2-y^2)$ simetrična je odnosu na koordinatne ravni (sl. 70), pa je



Sl. 70

$$P=8\iint_{S_1}dS=$$

$$=8\iint_{S_1}\sqrt{1+\left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2}\,dxdy=$$

$$=8\iint_{D_1}\frac{R}{\sqrt{R^2-(x^2+y^2)}}\,dxdy,$$

gdje je S_1 dio površi S u I oktantu.
Uvedimo polarne koordinate. Biće

$$P=8R\int_{\pi/4}^{\pi/2}d\varphi\int_0^R\frac{\rho\,d\rho}{\sqrt{R^2-\rho^2}}+8R\int_0^{\pi/4}d\varphi\int_{R\sqrt{\cos 2\varphi}}^{R}\frac{\rho\,d\rho}{\sqrt{R^2-\rho^2}}.$$

$$\left(\text{Može i ovako: } P=8R\int_0^{\pi/2}d\varphi\int_0^R\frac{\rho\,d\rho}{\sqrt{\rho^2-\rho^2}}-8R\int_0^{\pi/4}d\varphi\int_0^{R\sqrt{\cos\varphi}}\frac{\rho\,d\rho}{\sqrt{r^2-\rho^2}}.\right)$$

Dakle, $P=8R\displaystyle\int_0^{\pi/4}\left[-\sqrt{R^2-\rho^2}\right]_{R\sqrt{\cos 2\varphi}}^{R}d\varphi+8R\cdot\frac{\pi}{4}\left[-\sqrt{R^2-\rho^2}\right]_0^R$,

tj. $P=R^2(8\sqrt{2}-8+2\pi)$.

Izračunati površinu površi S, ako je:

250. S dio sfere $x^2+y^2+z^2=a^2$ unutar cilindra $x^2+y^2=ay$.

251. S dio cilindra $x^2=2z$ odsječenog ravnima $x-2y=0$, $y=2x$ i $x=2\sqrt{2}$.

252. S dio površi $y=x^2+z^2$ u I oktantu koji isijeca cilindar $x^2+z^2=1$.

253. S površ torusa $\vec{r}=(a+b\cos\theta)\cos\varphi\,\vec{i}+(a+b\cos\theta)\sin\varphi\,\vec{j}+b\sin\theta\,\vec{k}$.

Rj.

250. $P=4a^2\left(\dfrac{\pi}{2}-1\right)$.

251. $P=13$.

252. $P=\dfrac{(5\sqrt{5}-1)}{24}$.

253. $P=4ab\pi^2$ $\left(\text{Uzeti } P=\iint\limits_S dS=\iint\limits_D\left|\dfrac{\partial\vec{r}}{\partial\varphi}\times\dfrac{\partial\vec{r}}{\partial\theta}\right|d\varphi\,d\theta;\right.$

$\left.D:\ 0<\varphi\leqslant 2\pi,\ 0\leqslant\theta\leqslant 2\pi.\right)$.

Izračunati površinu površi S, ako je S površ $(x^2+y^2+z^2)^2=a^2(x^2+y^2)$.

Rj.

Smjenom $x=\rho\cos\varphi\sin\theta$, $y=\rho\sin\varphi\sin\theta$, $z=\rho\cos\theta$ dobija se jednačina $\rho=a\sin\theta$. Uvrštavajući ovu vrijednost $\rho=\rho(\varphi,\theta)$ u jednačine $x=x(\rho,\varphi,\theta)$, $y=y(\rho,\varphi,\theta)$, $z=z(\rho,\varphi,\theta)$ dobijaju se parametarske jednačine površi

$$x=a\sin^2\theta\cos\varphi=x(\varphi,\theta)$$

$$y=a\sin^2\theta\sin\varphi=y(\varphi,\theta)$$

$$z=\frac{a}{2}\sin 2\theta \quad =z(\varphi,\theta).$$

Za izračunavanje površine koristićemo vezu

$$\iint\limits_S dS=\iint\limits_D\sqrt{A^2+B^2+C^2}\,d\varphi\,d\theta, \text{ pri čemu je}$$

$$A=\frac{D(y,z)}{D(\varphi,\theta)},\quad B=\frac{D(z,x)}{D(\varphi,\theta)},\quad C=\frac{D(x,y)}{D(\varphi,\theta)}.$$

Biće:

$$A=a^2\sin^2\theta\cos\varphi\cos 2\theta$$

$$B=a^2\sin^2\theta\sin\varphi\cos 2\theta$$

$$C=-2a^2\cos\theta\sin^3\theta$$

i zatim

$$A^2+B^2+C^2=a^4\sin^4\theta,\ \sqrt{A^2+B^2+C^2}=a^2\sin^2\theta.$$

Sada je

$$P=a^2\int_0^{2\pi}d\varphi\int_0^{\pi}\sin^2\theta\,d\theta=2\pi a^2\int_0^{\pi}\sin^2\theta\,d\theta=\pi^2a^2.$$

(#) **Izračunati površinu površi S, ako je S površ (Vivanijevog) tijela**

$$V=\{(x, y, z): x^2+y^2+z^2\leqslant R^2, x^2+y^2\leqslant Rx\}.$$

Rj.

Tijelo je simetrično u odnosu na ravan $z=0$, pa je $P=2P_S+2P_C$, pri čemu je P_S površina gornjeg dijela sfere, a P_C površina gornjeg dijela cilindra. Biće

$$P_S=\iint_{S'}\sqrt{1+z_x'^2+z_y'^2}\,dxdy=R\iint_{S'}\frac{dxdy}{\sqrt{R^2-(x^2+y^2)}}.$$

Oblast S' je krug $x^2+y^2\leqslant Rx$. Uvodeći polarne koordinate dobija se

$$P_S=R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi\int_0^{R\cos\varphi}\frac{\rho\,d\rho}{\sqrt{R^2-\rho^2}}=2R^2\left(\frac{\pi}{2}-1\right).$$

Površinu cilindra tražimo pomoću krivolinijkog integrala: $P_C=\int_l zds$, pri čemu je l kružnica $x^2+y^2=Rx$, a $z=\sqrt{R^2-(x^2+y^2)}=\sqrt{R^2-Rx}$.

Ako jednačinu kružnice l napišemo u parametarskom obliku

$$x=\frac{R}{2}(1-\cos\varphi),\quad y=\frac{R}{2}\sin\varphi,\quad \text{dobiće se } ds=\frac{R}{2}d\varphi,\quad \varphi\in(0, 2\pi),\quad \text{i zatim}$$

$$P_C=\frac{R}{2}\int_0^{2\pi}\sqrt{R^2-\frac{R^2}{2}(1+\cos\varphi)}\,d\varphi=\frac{R^2}{2}\int_0^{2\pi}\sqrt{1-\cos^2\varphi}\,d\varphi=$$

$$=\frac{R^2}{2}\int_0^{2\pi}\sqrt{\sin^2\varphi}\,d\varphi=\frac{R^2}{2}\int_0^{2\pi}|\sin\varphi|\,d\varphi=$$

$$=\frac{R^2}{2}\int_0^{\pi}\sin\varphi\,d\varphi-\frac{R^2}{2}\int_{\pi}^{2\pi}\sin\varphi\,d\varphi=2R^2.$$

Slijedi

$$P=2R^2.$$

Stoksova formula

Dat je krivolinijski integral $\int_c P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz$

gdje je c kontura u prostoru.
Stoksova formula glasi:

$$\int_c P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz=\iint_S\begin{vmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}dS$$

površinski integral prve vrste

$$\int_c P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz=\iint_S\begin{vmatrix}dydz & dxdz & dxdy\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$$

površinski integral druge vrste

gdje je S površina u prostoru ograničena konturom c a $\vec{n}=(\cos\alpha, \cos\beta, \cos\gamma)$ jedinični vektor normale na površinu S.

$$\begin{vmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\cos\alpha+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\cos\beta+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\cos\gamma$$

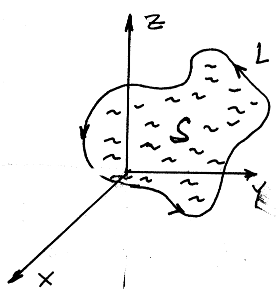
Vidimo da Stoksova formula povezuje krivolinijski integral druge vrste sa površinskim integralom prve i druge vrste.

Ranije smo spomenuli Greenovu formulu koja povezuje krivolinijski integral druge vrste sa dvostrukim integralom. Formula Gauss-Ostrogradski povezuje površinski integral druge vrste sa trostrukim integralom.

Integral $I=\int\limits_L (y^2+z^2)\,dx+(x^2+z^2)\,dy+(x^2+y^2)\,dz$

uzet po nekoj zatvorenoj konturi L, pretvoriti pomoću formule Stoksa u površinski integral nad površinom koju zatvara spomenuta kontura.

Rj.



$$\int\limits_L P\,dx+Q\,dy+R\,dz=\iint\limits_S \begin{vmatrix} dy\,dz & dx\,dz & dx\,dy\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$$

$R(x,y,z)=x^2+y^2$

$P(x,y,z)=y^2+z^2$

$Q(x,y,z)=x^2+z^2$

$\frac{\partial R}{\partial y}=2y \quad \frac{\partial Q}{\partial z}=2z$

$\frac{\partial R}{\partial x}=2x \quad \frac{\partial P}{\partial z}=2z$

$\frac{\partial Q}{\partial x}=2x \quad \frac{\partial P}{\partial y}=2y$

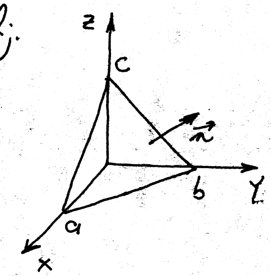
$$I=\iint\limits_S (2y-2z)\,dy\,dz-(2x-2z)\,dx\,dz+(2x-2y)\,dx\,dy=$$

$$=2\iint\limits_S (y-z)\,dy\,dz+(z-x)\,dx\,dz+(x-y)\,dx\,dy$$

Izračunati krivoliniski integral $-\int\limits_C y^2dx+z^2dy+x^2dz$

pri čemu je c kontura ΔABC gdje su tačke $A(a,0,0)$, $B(0,b,0)$ i $C(0,0,c)$, $a,b,c>0$.

Rj.



$$-\int\limits_C y^2dx+z^2dy+x^2dz \overset{\text{Stoksova formula}}{=\!=} -\iint\limits_S \begin{vmatrix} dy\,dz & dz\,dx & dx\,dy\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$$

$P=y^2,\quad Q=z^2,\quad R=x^2$

$\frac{\partial Q}{\partial x}=0,\quad \frac{\partial P}{\partial y}=2y,\quad \frac{\partial R}{\partial y}=0,\quad \frac{\partial Q}{\partial z}=2z$

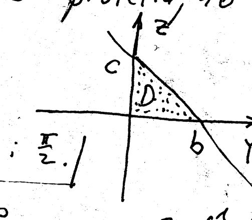
$\frac{\partial R}{\partial x}=2x \quad \frac{\partial P}{\partial z}=0$

$$\begin{vmatrix} dy\,dz & dz\,dx & dx\,dy\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=-2z\,dy\,dz-2x\,dz\,dx-2y\,dx\,dy$$

$$-\int\limits_C y^2dx+z^2dy+x^2dz=2\iint\limits_S (z\,dy\,dz+x\,dz\,dx+y\,dx\,dy)$$

S oblast ograničena ΔABC

Izračunajmo $\iint\limits_S z\,dy\,dz$. Površinu S projicirajmo na yOz ravan:



$\frac{y}{b}+\frac{z}{c}=1$

$\frac{y}{b}+\frac{z}{c}=1$

$cy+bz=bc$

$bz=bc-cy$

$z=c-\frac{c}{b}y=\frac{c}{b}(b-y)$

Ugao koji zaklapa vektor normale $\vec{n}$ na površ S je između 0 i $\frac{\pi}{2}$.

$D:\begin{cases}0\le y\le b\\ 0\le z\le \frac{c}{b}(b-y)\end{cases}$

$\vec{n}=\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$

$\cos\alpha\ge 0$

$$\iint\limits_S z\,dy\,dz=+\iint\limits_D z\,dy\,dz=\int_0^b\Big[\int_0^{\frac{c}{b}(b-y)} z\,dz\Big]dy=\int_0^b \frac{1}{2}z^2\Big|_0^{\frac{c}{b}(b-y)}dy=\frac{1}{2}\left(\frac{c}{b}\right)^2\int_0^b (b-y)^2dy$$

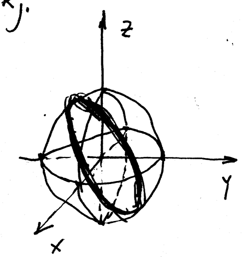
$$=\left|\begin{matrix} b-y=t & -dy=dt\\ y=0\Rightarrow t=b & dy=-dt\\ y=b\Rightarrow t=0 & \end{matrix}\right|=\frac{1}{2}\,\frac{c^2}{b^2}\int_0^b t^2dt=\frac{1}{2}\cdot\frac{c^2}{b^2}\,\frac{t^3}{3}\Big|_0^b=\frac{1}{2}\cdot\frac{bc^2}{3}$$

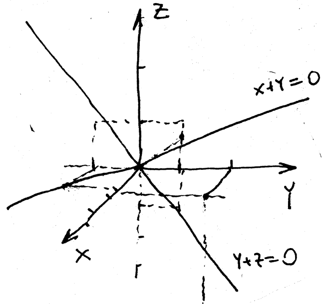
Analogno izračunamo $\iint\limits_S x\,dz\,dx=\frac{1}{2}\,\frac{a^2c}{3}$ i $\iint\limits_S y\,dx\,dy=\frac{1}{2}\cdot\frac{ab^2}{3}$ $\Rightarrow$ $I=\frac{ab^2+bc^2+ca^2}{3}$

Izračunati krivoliniski integral $\int_c y\,dx+z\,dy+x\,dz$

ako je c krug dobijen presjekom sfere $x^2+y^2+z^2=a^2$ i ravni $x+y+z=0$.

Rj.





$$\int_c y\,dx+z\,dy+x\,dz \overset{\text{Stoksova formula}}{=} \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

$\frac{\partial Q}{\partial x}=0 \quad \frac{\partial P}{\partial y}=1 \qquad P=y,\ Q=z,\ R=x$

S je površina ograničena krugom

$$\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\cos\alpha-\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)\cos\beta+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\cos\gamma$$

$\frac{\partial R}{\partial y}=0 \quad \frac{\partial Q}{\partial z}=1 \qquad \frac{\partial R}{\partial x}=1 \quad \frac{\partial P}{\partial z}=0$

$$\int_c y\,dx+z\,dy+x\,dz=\iint_S(-\cos\alpha-\cos\beta-\cos\gamma)\,dS$$

gdje je $\vec{n}_0=(\cos\alpha,\cos\beta,\cos\gamma)$ vektor (jedinični) normale na površinu S

$x+y+z=0$

$\vec{n}=(1,1,1)$ vektor normale na ravan $x+y+z=0$ (a time i na našu površinu S)

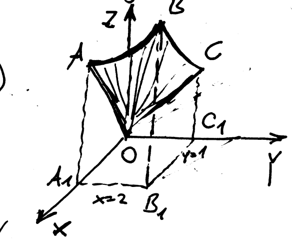
$|\vec{n}|=\sqrt{1+1+1}=\sqrt{3}$ $\qquad \vec{n}_0=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ ($\cos\alpha$, $\cos\beta$, $\cos\gamma$)

$$\iint_S(-\cos\alpha-\cos\beta-\cos\gamma)\,dS=\iint_S\left(-\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}\right)dS=-\frac{3}{\sqrt{3}}\iint_S dS$$

$\iint_S dS$ je površina oblasti S (S je krug poluprečnika a $\quad P_{kruga}=a^2\pi$)

$$\int_c y\,dx+z\,dy+x\,dz=-\frac{3}{\sqrt{3}}a^2\pi=-\sqrt{3}\,a^2\pi$$

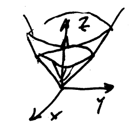
Uz pomoć formule Stoksa, izračunati krivoliniski integral $K=\oint_\ell e^x\,dx+z(x^2+y^2)^{\frac{3}{2}}dy+yz^3\,dz$ gdje je



ℓ-zakrivljena linija OCBAO (vidi sliku) dobijena presjekom površina

$z=\sqrt{x^2+y^2}$, $x=0$, $x=2$, $y=0$, $y=1$.

Rj.



$z=\sqrt{x^2+y^2}$ je čunj iznad xOy ravni

$x=0$, $x=2$ su ravni paralelne sa yOz ravni

$y=0$, $y=1$ su ravni paralelne sa xOz ravni

Stoksova formula glasi

$$\int_c P(x,y,z)\,dx+Q(x,y,z)\,dy+R(x,y,z)\,dz=\iint_S\begin{vmatrix} dy\,dz & dx\,dz & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

površinski integral druge vrste

$P=e^x,\ \frac{\partial P}{\partial y}=0,\ \frac{\partial P}{\partial z}=0$

$Q=z(x^2+y^2)^{\frac{3}{2}},\ \frac{\partial Q}{\partial x}=z\cdot\frac{3}{2}(x^2+y^2)^{\frac{1}{2}}\cdot 2x=3xz\sqrt{x^2+y^2},\ \frac{\partial Q}{\partial z}=(x^2+y^2)^{\frac{3}{2}}$

$R=yz^3,\ \frac{\partial R}{\partial x}=0,\ \frac{\partial R}{\partial y}=z^3$

$$K=\oint_\ell e^x\,dx+z(x^2+y^2)^{\frac{3}{2}}dy+yz^3\,dz=\left|\text{formula Stoksa}\right|=$$

$$=\iint_S\begin{vmatrix} dy\,dz & dx\,dz & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & z(x^2+y^2)^{\frac{3}{2}} & yz^3 \end{vmatrix}=\iint_S\Big(\underbrace{z^3-\underset{(\sqrt{x^2+y^2})^3}{\overset{\parallel}{(x^2+y^2)^{\frac{3}{2}}}}}_{=0}\Big)dy\,dz-(0-0)\,dx\,dz$$

$$+\left(3xz\sqrt{x^2+y^2}-0\right)dx\,dy=\iint_S 3xz\sqrt{x^2+y^2}\,dx\,dy$$

površinski integral II vrste

Tj. dobili smo $K = \iint_S 3xz\sqrt{x^2+y^2}\,dx\,dy$

Kako naša data kriva pravi površinu $S: z=\sqrt{x^2+y^2}$ u prvom oktantu imamo

$$K = \iint_S 3x\,(x^2+y^2)\,dx\,dy$$

Prisjetimo se kako se računa površinski integral II vrste npr. $I=\iint_S R(x,y,z)\,dx\,dy$. Neka je $\vec{n}$ vektor normale na površ S, neka je γ ugao koji $\vec{n}$ gradi sa z-osom, i neka je D projekcija površi S na xOy ravan. Tada

$$I=\iint_S R(x,y,z)\,dx\,dy = \pm\iint_D R(x,y,z(x,y))\,dx\,dy$$ gdje predznak ispred integrala zavisi od $\cos\gamma$ (za $\cos\gamma>0$ $+$, $\cos\gamma<0$ $-$)

Mi posmatramo vanjsku stranu površi, iz čega možemo zaključiti (sa slike) da je $\gamma\in(\frac{\pi}{2},\pi)$ pa je $\cos\gamma<0$.

Projekcija D površi S je data u sklopu zadatka (vidi sliku) ($\square A_1B_1C_1O$)

$$D:\begin{cases}0\le x\le 2\\ 0\le y\le 1\end{cases}\qquad K=\iint_S 3x(x^2+y^2)\,dx\,dy = -\iint_D 3x(x^2+y^2)\,dx\,dy =$$

$$= -3\int_0^1 dy\int_0^2 (x^3+xy^2)\,dx = -3\int_0^1\left(\frac{1}{4}x^4\Big|_0^2 + \frac{1}{2}x^2y^2\Big|_0^2\right)dy =$$

$$= -3\int_0^1 (4+2y^2)\,dy = -3\left(4y\Big|_0^1 + \frac{2}{3}y^3\Big|_0^1\right) = -12-2 = -14$$ traženo rješenje

Zadaci za vježbu

3894. Integral $\int_L (y^2+z^2)\,dx+(x^2+z^2)\,dy+(x^2+y^2)\,dz$, uzet po nekoj zatvorenoj konturi L, primenom Štoksove formule transformisati u integral po površini „razapetoj" nad tom konturom.

3895. Izračunati integral $\int_L x^2y^3\,dx+dy+z\,dz$ po krugu $x^2+y^2=R^2$, $z=0$, na dva načina: a) neposredno, i b) koristeći Štoksovu formulu, uzimajući za površinu S polusferu $z=+\sqrt{R^2-x^2-y^2}$. (Integracija po krugu u ravni xOy računa se u pozitivnom smeru obilaženja).

Rješenja

3894. $2\iint_S (x-y)\,dx\,dy+(y-z)\,dy\,dz+(z-x)\,dx\,dz.$

3895. $-\dfrac{\pi R^6}{8}.$

Formula Gauss-Ostrogradski

Ova formula daje vezu između površinskog integrala druge vrste i trostrukog integrala.

$$\iint P(x,y,z)\,dy\,dz + Q(x,y,z)\,dx\,dz + R(x,y,z)\,dx\,dy =$$

$$= \iiint\limits_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

gdje je Ω oblast u prostoru ograničena datom površinom S (S je zatvorena površina).

(1.) Izračunati $\iint\limits_S xy\,dx\,dy + yz\,dy\,dz + zx\,dz\,dx$ gdje je S bilo koja zatvorena površ.

Rj.

$$\iint\limits_S yz\,dy\,dz + zx\,dx\,dz + xy\,dx\,dy = \iint\limits_S P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy \overset{\text{Formula Gauss-Ostr.}}{=} \iiint\limits_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

gdje je Ω oblast u prostoru ograničena datom površinom S, $\frac{\partial P}{\partial x} = 0$; $\frac{\partial Q}{\partial y} = 0$, $\frac{\partial R}{\partial z} = 0$

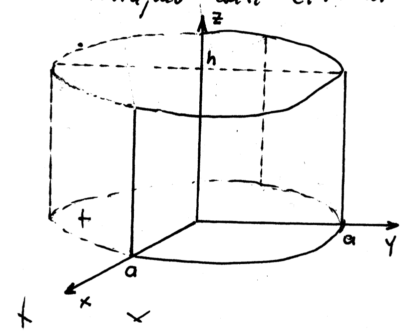
$$\iint\limits_S yz\,dy\,dz + zx\,dx\,dz + xy\,dx\,dy = \iiint\limits_{\Omega} 0\,dx\,dy\,dz =$$

$$\Omega : \begin{cases} a \le x \le b \\ c \le y \le d \\ e \le z \le f \end{cases} \qquad = \int_a^b dx \int_c^d dy \int_e^f 0\,dz = 0$$

(#) Uz pomoć formule Gauss-Ostrogradski izračunati površinski integral $I = \oiint\limits_S 4x^3\,dy\,dz + 4y^3\,dx\,dz - 6z^4\,dx\,dy$

gdje je S vanjska strana cilindra $x^2+y^2=a^2$ koji se nalazi između ravni $z=0$ i $z=h$.

Rj. Skicirajmo dati cilindar



Prisjetimo se formule Gauss-Ostrogradski

$$\oiint\limits_S P(x,y,z)\,dy\,dz + Q(x,y,z)\,dx\,dz + R(x,y,z)\,dx\,dy = \iiint\limits_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

Ω – unutrašnjost oblasti S

$P(x,y,z) = 4x^3$ $\quad \frac{\partial P}{\partial x} = 12x^2$

$Q(x,y,z) = 4y^3$ $\quad \frac{\partial Q}{\partial y} = 12y^2$

$R(x,y,z) = 6z^4$ $\quad \frac{\partial R}{\partial z} = 24z^3$

$$\oiint\limits_S 4x^3\,dy\,dz + 4y^3\,dx\,dz - 6z^4\,dx\,dy = 12\iiint\limits_{\Omega}(x^2+y^2-2z^3)\,dx\,dy\,dz =$$

$$= \left| \begin{array}{ll} \text{uvedimo cilindrične koordinate} & \\ x = r\cos\varphi & \Omega \xrightarrow{\text{transformiše}} \Omega' \\ y = r\sin\varphi & 0 \le r \le a \\ z = z & 0 \le \varphi \le 2\pi \\ dx\,dy\,dz = r\,dr\,d\varphi\,dz & 0 \le z \le h \\ x^2+y^2 = \ldots = r^2 & \end{array} \right| = 12\iiint\limits_{\Omega'}(r^2-2z^3)\,r\,dr\,d\varphi\,dz =$$

$$= 12\int_0^{2\pi} d\varphi \int_0^a dr \int_0^h (r^3 - 2rz^3)\,dz = 12\int_0^{2\pi} d\varphi \int_0^a \left(r^3 z\Big|_0^h - 2r\cdot\frac{1}{4}z^4\Big|_0^h\right) dr$$

$$= 12\int_0^{2\pi} d\varphi \int_0^a \left(r^3 h - \frac{1}{2} r h^4\right) dr = 12\,\varphi\Big|_0^{2\pi}\left(h\frac{1}{4}r^4\Big|_0^a - \frac{1}{2}h^4\cdot\frac{1}{2}r^2\Big|_0^a\right) =$$

$$= 24\pi\cdot\frac{1}{4}h\,(a^4 - h^3a^2) = 6\pi h a^2 (a^2 - h^3)$$ traženo rješenje

(#) Površinski integral po zatvorenoj površini pretvoriti uz pomoć formule Ostrogradskog u trostruki integral po zapremini tijela, koje je ograničeno spomenutom površinom

$$\iint_S \sqrt{x^2+y^2+z^2}\,\left[\cos(\vec{n},x)+\cos(\vec{n},y)+\cos(\vec{n},z)\right]dS$$

gdje je $\vec{n}$ vanjska normala na površinu S.

Rj. $\cos(\vec{n},x)$ je kosinus ugla između normale i x-ose. $\cos(\vec{n},y)$ i $\cos(\vec{n},z)$ je kosinus ugla između normale na površinu S i y-ose i z-ose redom.

Uvedimo oznake $\cos(\vec{n},x)=\cos\alpha$, $\cos(\vec{n},y)=\cos\beta$ i $\cos(\vec{n},z)=\cos\gamma$.

Prema formuli Stoksa znamo da je

$$dy\,dz=dS\cos\alpha$$
$$dz\,dx=dS\cos\beta$$
$$dx\,dy=dS\cos\gamma$$

$$I=\iint_S \sqrt{x^2+y^2+z^2}\left(\cos(\vec{n},x)+\cos(\vec{n},y)+\cos(\vec{n},z)\right]dS=$$

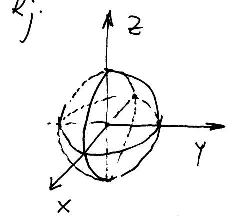
$$=\iint_S \sqrt{x^2+y^2+z^2}\,(dy\,dz+dz\,dx+dx\,dy)$$

$$\iint_S P\,dy\,dz+Q\,dx\,dz+R\,dx\,dy=\iiint_\Omega\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz$$

$$\frac{\partial P}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2+z^2}} \qquad I=\iiint_\Omega \frac{x+y+z}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz$$

(#) Izračunati $\iint_S x^3\,dy\,dz+y^3\,dz\,dx+z^3\,dx\,dy$ gdje je S - vanjski dio sfere $x^2+y^2+z^2=R^2$.

Rj.



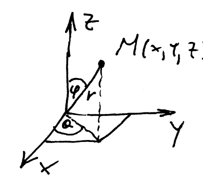
$$I=\iint_S x^3\,dy\,dz+y^3\,dz\,dx+z^3\,dx\,dy \underset{\text{Gauss-Ostrogradski}}{\overset{\text{formula}}{=}}$$

$$=\iiint_\Omega\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz$$

$$P=x^3,\quad \frac{\partial P}{\partial x}=3x^2,\quad Q=y^3,\quad \frac{\partial Q}{\partial y}=3y^2,\quad R=z^3,\quad \frac{\partial R}{\partial z}=3z^2$$

$$I=\iiint_\Omega (3x^2+3y^2+3z^2)\,dx\,dy\,dz \qquad \Omega: x^2+y^2+z^2\le R^2$$

Uvedimo sferne koordinate:



$$x=r\sin\varphi\cos\theta$$
$$y=r\sin\varphi\sin\theta$$
$$z=r\cos\varphi$$

$$\Omega':\begin{cases}0\le r\le R\\ 0\le\theta\le 2\pi\\ 0\le\varphi\le\pi\\ dx\,dy\,dz=r^2\sin\varphi\,d\varphi\,d\theta\,dr\end{cases}$$

$$x^2+y^2+z^2=r^2$$

$$I=3\iiint_{\Omega'} r^2 r^2\sin\varphi\,d\varphi\,d\theta\,dr=3\int_0^R r^4\,dr\int_0^{2\pi}d\theta\int_0^\pi \sin\varphi\,d\varphi=$$

$$=3\cdot\frac{1}{5}r^5\Big|_0^R\cdot\theta\Big|_0^{2\pi}\cdot(-\cos\varphi)\Big|_0^\pi=\frac{3}{5}\cdot R^5\cdot 2\pi\cdot 2=\frac{12}{5}R^5\pi$$

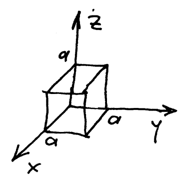
(#) Izračunati $\iint_S x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy$ gdje je S - vanjska strana kocke $0 \le x \le a,\ 0 \le y \le a,\ 0 \le z \le a$.

Rj.:

formula Gauss-Ostr.

$$\iint_S P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy = \iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$$

$$\frac{\partial P}{\partial x} = 2x, \quad \frac{\partial Q}{\partial y} = 2y, \quad \frac{\partial R}{\partial z} = 2z$$



$$\Omega : \begin{cases} 0 \le x \le a \\ 0 \le y \le a \\ 0 \le z \le a \end{cases}$$

Prema tome:

$$\iint_S x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy =$$

$$= \iiint_\Omega (2x + 2y + 2z)\,dx\,dy\,dz = 2\int_0^a dx \int_0^a dy \int_0^a (x+y+z)\,dz =$$

$$= 2\int_0^a dx \int_0^a \left(xz\Big|_0^a + yz\Big|_0^a + \frac{1}{2}z^2\Big|_0^a \right) dy = 2\int_0^a dx \int_0^a \left(ax + ay + \frac{1}{2}a^2\right) dy =$$

$$2a\int_0^a \left(xy\Big|_0^a + \frac{1}{2}y^2\Big|_0^a + \frac{1}{2}ay\Big|_0^a \right) dx = 2a\int_0^a \left(ax + \frac{1}{2}a^2 + \frac{1}{2}a^2\right) dx = 2a^2\int_0^a (x+a)\,dx =$$

$$= 2a^2\left(\frac{1}{2}a^2 + a^2\right) = 3a^4$$

Zadaci za vježbu

3896. Površinski integral $\iint_S x^2\,dy\,dz + y^2\,dx\,dz + z^2\,dx\,dy$, uzet po zatvorenoj površini S, primenom formule Ostrogradskog, transformisati u trojni integral po zapremini ograničenoj tom površinom (Integral se računa po spoljnoj strani površine S).

3897. Površinski integral $\iint_S \sqrt{x^2 + y^2 + z^2}\{\cos(N,x) + \cos(N,y) + \cos(N,z)\}\,d\sigma$ po zatvorenoj površini S, primenom formule Ostrogradskog transformisati u trojni integral po zapremini ograničenoj tom površinom, pri čemu je N spoljna normala površine S.

3898. Izračunati integral u prethodnom zadatku ako je S sfera poluprečnika R sa centrom u koordinatnom početku.

3899. Izračunati integral

$$\iint_S [x^3\cos(N,\ x) + y^3\cos(N,\ y) + z^3\cos(N,\ z)]\,d\sigma,$$

u kojem je S — sfera poluprečnika R sa centrom u koordinatnom početku, a N — spoljna normala.

3900. Izračunati integral u zadacima 3891—3863 primenom formule Ostrogradskog.

Rješenja

3896. $2\iiint_\Omega (x+y+z)\,dx\,dy\,dz.$

3897. $\iiint_\Omega \frac{x+y+z}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz.$ **3898.** 0. **3899.** $\frac{12}{5}\pi R^5.$

Vektorska teorija polja

Skalarno polje je f-ja $u = f(T) = f(x, y, z)$ u oblasti prostora ili na površi (na primjer, temperatura u svakoj tački prostora, nadmorska visina tačke i dr.) Skalarno polje se predstavlja nivoskim površima tj. površima s jednačinom $u = c \cdot f(T) = c \cdot f(x, y, z)$ (gdje je c-konstanta) i u ima neprekidne parcijalne izvode koji se ne anuliraju istovremeno).

Na primjer $u = x^2 + y^2 + z^2$ je skalarno polje.

Ranije smo spomenuli da je gradijent f-je $u = f(x, y, z)$, date u nekoj oblasti prostora, vektor čije su projekcije na ose Dekartovog koordinatnog sistema $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ i $\frac{\partial u}{\partial z}$. Označava se simbolom

$$\operatorname{grad} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

Izvod u pravcu gradijenta u datoj tački dostiže najveću vrijednost jednaku $|\operatorname{grad} u| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}$ tj. pravac gradijenta je pravac najbržeg rasta f-je.

Vektorsko polje je oblast prostora u čijoj je svakoj tački definisan vektor.

$\vec{v} = (V_x, V_y, V_z) = V_x \vec{i} + V_y \vec{j} + V_z \vec{k}$ gdje su V_x, V_y, V_z skalarna polja.

Na primjer $\vec{v} = (y^2 + z^2)\vec{i} + x^2 \vec{j} + xyz^2 \vec{k}$ je vektorsko polje.

Nabla operator (∇ operator ili Hamiltonov operator) je diferencijalni operator oblika $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ gdje su $\vec{i}, \vec{j}, \vec{k}$ jedinični ortogonalni vektori.

Ako je $u = f(x, y, z)$ skalarna f-ja biće

$$\nabla \cdot f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} = \operatorname{grad} f$$

Ako je $\vec{v} = (V_x, V_y, V_z)$ vektorska f-ja onda je $\nabla \vec{v} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

Važne osobine vektorskog polja su divergencija i rotor vektorskog polja

$$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \quad (\text{skalarni proizvod } \nabla \text{ i } \vec{v})$$

$$\operatorname{rot} \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \quad (\text{vektorski proizvod } \nabla \text{ i } \vec{v})$$

Ako je $\operatorname{div} \vec{v} = 0$ tada kažemo da je $\vec{v}$ solenoidno polje.
Ako je $\operatorname{rot} \vec{v} = \vec{0}$ tada kažemo da je $\vec{v}$ potencijalno polje.
F-ju u za koju vrijedi da je $\vec{v} = \operatorname{grad} u$ zovemo potencijalom polja $\vec{v}$.

relacija $u(x, y, z) = C$ gdje je C konstanta, predstavlja površ koju zovemo ekviskalarna površ (nivo površ) skalarnog polja

Naći veličinu i pravac gradijenta skalarnog polja: a) $u = x^2+y^2+z^2$ u tački $T(2,-2,1)$ b) $u = xyz$ u tački $T(1,2,3)$.

Rj. a)

$$\text{grad}\,u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

$$\text{grad}\,u = (2x, 2y, 2z) \Rightarrow \text{grad}\,u(T) = (4,-4,2)$$

$$|\text{grad}\,u| = \sqrt{16+16+4} = 6 \quad \text{veličina gradijenta}$$

$$\frac{\text{grad}\,u(T)}{|\text{grad}\,u(T)|} = \left(\frac{4}{6}, -\frac{4}{6}, \frac{2}{6}\right) = \Big(\underbrace{\tfrac{2}{3}}_{\cos\alpha}, \underbrace{-\tfrac{2}{3}}_{\cos\beta}, \underbrace{\tfrac{1}{3}}_{\cos\gamma}\Big) \quad \text{jedinični vektor pravca gradijenta}$$

$\alpha = \arccos\frac{2}{3}$

$\beta = \arccos\left(-\frac{2}{3}\right)$ $\quad\gamma = \arccos\frac{1}{3}$

b) $|\text{grad}\,u(T)| = 7 \qquad \dfrac{\text{grad}\,u(T)}{|\text{grad}\,u(T)|} = \left(\dfrac{6}{7}, \dfrac{3}{7}, \dfrac{2}{7}\right)$

Dato je skalarno polje $u = x^3+y^3+z^3-3xyz$. U kojim tačkama je a) $\text{grad}\,u = \vec{0}$ b) $\vec{k}\cdot\text{grad}\,u = 0$.

Rj.

a) $\text{grad}\,u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$

$$\text{grad}\,u = (3x^2-3yz, 3y^2-3xz, 3z^2-3xy)$$

$$\text{grad}\,u = \vec{0} \Rightarrow \begin{array}{l} 3x^2-3yz=0 \\ 3y^2-3xz=0 \\ 3z^2-3xy=0 \end{array} \qquad \begin{array}{ll} x^2-yz=0 & (I) \\ y^2-xz=0 & (II) \\ z^2-xy=0 & (III) \end{array}$$

Trivijalno rješenje sistema je $x=0$, $y=0$, $z=0$.

Ako pomnožimo (I) sa x ($x\neq0$), (II) sa y ($y\neq0$) i (III) sa z ($z\neq0$) dobijamo

$$\begin{array}{l} x^3-xyz=0 \\ y^3-xyz=0 \\ z^3-xyz=0 \end{array} \qquad \begin{array}{l} xyz=x^3 \\ xyz=y^3 \\ xyz=z^3 \end{array} \qquad \begin{array}{l} x^3=y^3=z^3 \\ x=y=z \end{array}$$

Ako ovu zadnju jednakost napišemo u obliku $\dfrac{x-0}{1} = \dfrac{y-0}{1} = \dfrac{z-0}{1}$ (prava u prostoru) vidimo da je $\text{grad}\,u = \vec{0}$ za sve tačke ove prave.

b) $\text{grad}\,u = (3x^2-3yz)\vec{i} + (3y^2-3xz)\vec{j} + (3z^2-3xy)\vec{k}$

$\vec{k}\cdot\text{grad}\,u = 3z^2-3xy = 0$

$\vec{k}\cdot\text{grad}\,u = 0$ je za sve tačke krive $z^2-xy=0$

Odrediti ugao kojeg zatvaraju gradijenti polja $z=\sqrt{x^2+y^2}$ i $u=x-3y+\sqrt{3xy}$ u tački $A(3,4)$.

Rj. Gradijent f-je $z=f(x,y)$ se računa po formuli

$$\operatorname{grad} z=\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right).$$

$$\operatorname{grad} z=\left(\frac{1}{2\sqrt{x^2+y^2}}\cdot 2x,\ \frac{1}{2\sqrt{x^2+y^2}}\cdot 2y\right)=\left(\frac{x}{\sqrt{x^2+y^2}},\ \frac{y}{\sqrt{x^2+y^2}}\right)$$

$u=x-3y+\sqrt{3xy}$, $\quad \frac{\partial u}{\partial x}=1+\frac{1}{2\sqrt{3xy}}\cdot 3y=1+\frac{3y}{2\sqrt{3xy}}$

$$\frac{\partial u}{\partial y}=-3+\frac{3x}{2\sqrt{3xy}}$$

$$\operatorname{grad} u=\left(1+\frac{3y}{2\sqrt{3xy}},\ -3+\frac{3x}{2\sqrt{3xy}}\right).$$

$A(3,4)$, $\operatorname{grad} z(A)=\left(\frac{3}{\sqrt{9+16}},\ \frac{4}{\sqrt{9+16}}\right)=\left(\frac{3}{5},\frac{4}{5}\right)=\frac{3}{5}\vec{i}+\frac{4}{5}\vec{j}$,

$$\operatorname{grad} u(A)=\left(1+\frac{12}{2\sqrt{36}},\ -3+\frac{9}{2\sqrt{36}}\right)=\left(1+1,\ -3+\frac{\cancel{9}^{3}}{\cancel{12}_{4}}\right)=$$

$$=\left(2,-\frac{9}{4}\right)=2\vec{i}-\frac{9}{4}\vec{j}$$

$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cdot\cos\varphi \quad\Rightarrow\quad \cos\varphi=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

U našem slučaju $\vec{a}=\left(\frac{3}{5},\frac{4}{5}\right)$, $\vec{b}=\left(2,-\frac{9}{4}\right)$

$$\vec{a}\cdot\vec{b}=\frac{3}{5}\cdot 2+\frac{4}{5}\cdot\left(-\frac{9}{4}\right)=\frac{6\cdot 4}{5\cdot 4}-\frac{36}{20}=\frac{24-36}{20}=\frac{-12}{20}=\frac{-3}{5}$$

$$|\vec{a}|=\sqrt{\frac{9}{25}+\frac{16}{25}}=1,\quad |\vec{b}|=\sqrt{4+\frac{81}{16}}=\sqrt{\frac{64+81}{16}}=\frac{\sqrt{145}}{4}$$

$$\cos\varphi=\frac{-\frac{3}{5}}{1\cdot\frac{\sqrt{145}}{4}}=\frac{-12}{\sqrt{145}}\quad\Rightarrow\quad \varphi=\arccos\left(\frac{-12}{\sqrt{145}}\right)$$ ugao kojeg zatvaraju gradijenti polja

Odrediti divergenciju i rotor vektorskih polja

a) $\vec{v}=(y^2+z^2)\vec{i}+(z^2+x^2)\vec{j}+(x^2+y^2)\vec{k}$

b) $\vec{v}=x^2yz\,\vec{i}+xy^2z\,\vec{j}+xyz^2\,\vec{k}$

Rj. a) $\operatorname{div}\vec{v}=\nabla\cdot\vec{v}=\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}$, $\quad (\vec{v}=(V_x,V_y,V_z))$

$V_x=y^2+z^2$, $\quad \frac{\partial V_x}{\partial x}=0$

$V_y=z^2+x^2$, $\quad \frac{\partial V_y}{\partial y}=0$

$V_z=x^2+y^2$, $\quad \frac{\partial V_z}{\partial z}=0$

$\operatorname{div}\vec{v}=0+0+0=0$ divergencija vektorskog polja

Kako je $\operatorname{div}\vec{v}=0$ to je polje $\vec{v}$ solenoidno

$$\operatorname{rot}\vec{v}=\nabla\times\vec{v}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\times(V_x,V_y,V_z)=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ V_x&V_y&V_z\end{vmatrix}$$

$\frac{\partial V_z}{\partial y}=2y \qquad \frac{\partial V_y}{\partial z}=2z$

$\frac{\partial V_z}{\partial x}=2x \qquad \frac{\partial V_x}{\partial z}=2z$

$\frac{\partial V_y}{\partial x}=2x \qquad \frac{\partial V_x}{\partial y}=2y$

$$\operatorname{rot}\vec{v}=(2y-2z)\vec{i}-(2x-2z)\vec{j}+(2x-2y)\vec{k}=$$

$$=(2y-2z,\ 2z-2x,\ 2x-2y)$$ rotor vektorskog polja

Kako je $\operatorname{rot}\vec{v}\neq 0$ to polje nije potencijalno polje.

b) URADITI ZA VJEŽBU

Rj. $\operatorname{div}\vec{v}=6xyz$

$$\operatorname{rot}\vec{v}=(yx^2-yz^2)\vec{j}+(zy^2-zx^2)\vec{k}+(xz^2-xy^2)\vec{i}.$$

Izračunati ∇u ako je $u=f(r)$, $\vec{a}=(x,y,z)$ je vektor položaja tačke $M(x,y,z)$ i $r=|\vec{a}|$.

Rj. Da li je u vektorska ili skalarna f-ja?

$$r=|\vec{a}|=\sqrt{x^2+y^2+z^2}$$

$u=f(\sqrt{x^2+y^2+z^2})$ je skalarna f-ja

$$\nabla u = \frac{\partial u}{\partial x}\vec{i}+\frac{\partial u}{\partial y}\vec{j}+\frac{\partial u}{\partial z}\vec{k}, \quad u=f(r)$$

$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x}=f'_r\cdot\left(\sqrt{x^2+y^2+z^2}\right)'_x=f'_r\cdot\frac{2x}{2\sqrt{x^2+y^2+z^2}}=f'_r\cdot\frac{x}{r}$$

$$\frac{\partial u}{\partial y}=\frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y}=f'_r\cdot\left(\sqrt{x^2+y^2+z^2}\right)'_y=f'_r\cdot\frac{2y}{2\sqrt{x^2+y^2+z^2}}=f'_r\cdot\frac{y}{r}$$

$$\frac{\partial u}{\partial z}=\frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z}=f'_r\cdot\left(\sqrt{x^2+y^2+z^2}\right)'_z=f'_r\cdot\frac{2z}{2\sqrt{x^2+y^2+z^2}}=f'_r\cdot\frac{z}{r}$$

$$\nabla u=\left(f'_r\cdot\frac{x}{\sqrt{x^2+y^2+z^2}},\ f'_r\cdot\frac{y}{\sqrt{x^2+y^2+z^2}},\ f'_r\cdot\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$=\frac{f'_r}{r}(x\vec{i}+y\vec{j}+z\vec{k})=f'_r\cdot\frac{\vec{a}}{r}$$

Iskoristiti prethodni zadatak i izračunati $\nabla\frac{1}{r}$.

Rj. Ako stavimo $f(r)=\frac{1}{r}$ u prethodni zadatak dobijamo:

$$f'_r=\left(\frac{1}{r}\right)'_r=\frac{-1}{r^2}$$

$$\nabla u=\nabla\frac{1}{r}=-\frac{1}{r^2}\cdot\frac{\vec{a}}{r}=\frac{-1}{r^3}\vec{a}$$

Dokazati da je vektorsko polje potencijalno i naći njegov potencijal:

$$\vec{v}=2x(y^2+z^2)\vec{i}+2y(x^2+z^2)\vec{j}+2z(x^2+y^2)\vec{k}.$$

Rj. Vektorsko polje $\vec{v}$ je potencijalno ako je $rot\,\vec{v}=\vec{0}$.
Rotor vektorskog polja $rot\,\vec{v}$ se računa . .

$$rot\,\vec{v}=\nabla\times\vec{v}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ V_x&V_y&V_z\end{vmatrix}$$

(vektorski proizvod Nabla (∇) operatora i vektorskog polja $\vec{v}$)

$V_x=2x(y^2+z^2)$, $V_y=2y(x^2+z^2)$, $V_z=2z(x^2+y^2)$

$\frac{\partial V_x}{\partial y}=4xy$, $\frac{\partial V_y}{\partial x}=4xy$, $\frac{\partial V_z}{\partial x}=4xz$

$\frac{\partial V_x}{\partial z}=4xz$, $\frac{\partial V_y}{\partial z}=4yz$, $\frac{\partial V_z}{\partial y}=4yz$

$$rot\,\vec{v}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ V_x&V_y&V_z\end{vmatrix}=\vec{i}(4yz-4yz)-\vec{j}(4xz-4xz)+\vec{k}(4xy-4xy)$$

$$=(0,0,0)=\vec{0}$$ vektorsko polje je potencijalno

Potencijal polja $\vec{v}$ je f-ja u za koju vrijedi $\vec{v}=grad\,u$.

$$grad\,u=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)$$

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=2x(y^2+z^2)\\ \frac{\partial u}{\partial y}&=2y(x^2+z^2)\\ \frac{\partial u}{\partial z}&=2z(x^2+y^2)\end{aligned}\right\}\ldots(1)$$

$u=u(x,y,z)$

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$$

$$u=\int 2x(y^2+z^2)\,dx+\varphi(y,z)$$

$$u=x^2(y^2+z^2)+\varphi(y,z)$$

$$\left.\begin{aligned}\frac{\partial u}{\partial y}&=2yx^2+\varphi'_y\\ \frac{\partial u}{\partial z}&=2x^2z+\varphi'_z\end{aligned}\right\}\ldots(2)$$

(1) i (2) $\Rightarrow$ $\varphi'_y=2yz^2$, $\varphi'_z=2zy^2$ ... (*)

Odredimo f-ju φ: $\varphi=\int 2yz^2\,dy+\psi(z)$

$$\varphi=y^2z^2+\psi(z)$$

$$\varphi'=2y^2z+\psi' \ldots (**)$$

(*) i (**) $\Rightarrow$ $\psi'_z=0$ $\Rightarrow$ $\psi(z)=C$

$\Rightarrow \varphi=y^2z^2+C \Rightarrow u=x^2y^2+x^2z^2+y^2z^2+C$

Potencijal vektorskog polja je $u=x^2y^2+x^2z^2+y^2z^2+C$

(#) Odrediti konstante a, b i c tako da vektorsko polje $\vec{v} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ bude potencijalno i naći njegov potencijal.

Rj. Ako je $\text{rot}\,\vec{v} = \vec{0}$ tada je vektorsko polje $\vec{v}$ potencijalno.

$$\text{rot}\,\vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$V_x = x+2y+az$
$V_y = bx-3y-z$
$V_z = 4x+cy+2z$

$\frac{\partial V_z}{\partial y} = c$ $\quad \frac{\partial V_y}{\partial z} = -1$ $\quad \frac{\partial V_y}{\partial x} = b$

$\frac{\partial V_z}{\partial x} = 4$ $\quad \frac{\partial V_x}{\partial z} = a$ $\quad \frac{\partial V_x}{\partial y} = 2$

$$\text{rot}\,\vec{v} = (c+1)\vec{i} - (4-a)\vec{j} + (b-2)\vec{k} = (c+1,\ a-4,\ b-2)$$

Za vrijednosti $a=4$, $b=2$ i $c=-1$ vektorsko polje $\vec{v}$ je potencijalno polje.

$\vec{v} = (x+2y+4z,\ 2x-3y-z,\ 4x-y+2z)$

Potencijal polja $\vec{v}$ je f-ja u koja zavisi od 3 promjenjive $u = u(x,y,z)$ i za koju vrijedi $\vec{v} = \text{grad}\,u$.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

Nađimo f-ju u.

$$\frac{\partial u}{\partial x} = x+2y+4z \qquad \frac{\partial u}{\partial y} = 2x-3y-z \qquad \frac{\partial u}{\partial z} = 4x-y+2z \quad \ldots (*)$$

$$u = \int (x+2y+4z)\,dx + \varphi(y,z)$$

$$u = \frac{1}{2}x^2 + (2y+4z)x + \varphi(y,z) \Rightarrow \frac{\partial u}{\partial y} = 2x + \varphi'_y, \quad \frac{\partial u}{\partial z} = 4x + \varphi'_z \quad \overset{(*)}{\Longrightarrow}$$

$\overset{(*)}{\Rightarrow}$ $\varphi'_y = -3y - z$; $\varphi'_z = -y+2z \quad \ldots (**)$ Odredimo f-ju φ.

$$\varphi = \int (-3y-z)\,dy + \psi(z) = -\frac{3}{2}y^2 - yz + \psi(z)$$

$$\varphi'_z = -y + \psi'_z \overset{(**)}{\Longrightarrow} \psi'_z = 2z \Rightarrow \psi(z) = \int 2z\,dz = z^2 + c$$

$$\varphi(y,z) = -\frac{3}{2}y^2 - yz + z^2 + c \Rightarrow u = \frac{1}{2}x^2 + 2xy + 4xz - \frac{3}{2}y^2 - yz + z^2 + c$$

(#) Dato je vektorsko polje $\vec{A} = (e^x z - 2xy,\ 1-x^2,\ e^x + z)$. Pokazati da je polje $\vec{A}$ potencijalno i odrediti mu potencijal. Izračunati integral $\int_L \vec{A}\cdot d\vec{r}$ gdje je L duž PQ, $P(0,1,-1)$, $Q(2,3,0)$ orijentisana od tačke P prema tački Q.

Rj. Ako je rotor vektorskog polja $\vec{A}$ jednak $\vec{0}$ ($\text{rot}\,\vec{A} = \vec{0}$), tada za $\vec{A}$ kažemo da je potencijalno polje.

$$\text{rot}\,\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x z - 2xy & 1-x^2 & e^x + z \end{vmatrix}$$

$$= (0-0)\vec{i} - (e^x - e^x)\vec{j} + (-2x+2x)\vec{k} = (0,0,0) \Rightarrow A \text{ je potencijalno polje}$$

F-ju $u = u(x,y,z)$ za koju vrijedi da je $\vec{A} = \text{grad}\,u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$ zovemo potencijal polja $\vec{A}$. $\vec{A} = (e^x z - 2xy,\ 1-x^2,\ e^x + z)$

$$u = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$u = \int (e^x z - 2xy)\,dx + \varphi(y,z)$$

$$u = e^x z - x^2 y + \varphi(y,z)$$

$\frac{\partial u}{\partial y} = -x^2 + \varphi'_y$
$\frac{\partial u}{\partial y} = 1 - x^2$

$\varphi'_y = 1$

$\varphi(y,z) = y + \psi(z) \quad \ldots (1)$

$\frac{\partial u}{\partial z} = e^x + \varphi'_z$
$\frac{\partial u}{\partial z} = e^x + z$

$\varphi'_z = z \Rightarrow \varphi(y,z) = \frac{z^2}{2} + \psi(y) \quad \ldots (2)$

(1) i (2) $\Rightarrow \varphi(y,z) = y + \frac{z^2}{2}$

Potencijal vektorskog polja $\vec{A}$ je $u = e^x z - x^2 y + y + \frac{z^2}{2} + c$

$\int_L \vec{A}\cdot d\vec{r}$ zovemo cirkulacija vektorskog polja $\vec{A}$ duž krive L

$$C = \int \vec{A}\,d\vec{r} = \int V_x dx + V_y dy + V_z dz \quad \text{gdje je } \vec{A} = (V_x, V_y, V_z),\ d\vec{r} = (dx, dy, dz).$$

$$C=\int_L (e^x z-2xy)dx+(1-x^2)dy+(e^x+z)dz$$

ovo je krivolinijski integral druge vrste po krivoj duži u prostoru

Prisjetimo se, ako je c kriva u ravni opisana parametarskim jednačinama $x=\eta(t)$, $y=\mu(t)$ gdje je $t_1 \le t \le t_2$ tada krivolinijski integral se računa

$$\int_c P(x,y)dx+Q(x,y)dy=\int_{t_1}^{t_2}\Big(P(\eta(t),\mu(t))\eta'(t)+Q(\eta(t),\mu(t))\mu'(t)\Big)dt$$

Postavimo pravu kroz dvije date tačke $P(0,1,-1)$ i $Q(2,3,0)$.

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$ jednačina prave kroz dvije tačke

$P(\overset{x_1}{0},\overset{y_1}{1},\overset{z_1}{-1})$, $Q(\overset{x_2}{2},\overset{y_2}{3},\overset{z_2}{0})$

$$\frac{x-0}{2}=\frac{y-1}{2}=\frac{z+1}{1}\quad(=t)$$

$$L:\begin{cases}x=2t & dx=2dt\\ y=2t+1 & dy=2dt\\ z=t-1 & dz=dt\\ 0\le t\le 1\end{cases}$$

$$C=\int_0^1\Big(2\cdot\big(e^{2t}(t-1)-2\cdot 2t\cdot(2t+1)\big)+(1-4t^2)\cdot 2+\big(e^{2t}+(t-1)\big)\Big)dt$$

$$=\int_0^1\left(2e^{2t}t-2e^{2t}-16t^2-8t+2-8t^2+e^{2t}+t-1\right)dt$$

$$=\int_0^1 2e^{2t}t\,dt-\int_0^1 e^{2t}dt-24\int_0^1 t^2dt+\int_0^1(-7t+1)dt=\dots=-\frac{19}{2}$$

$$2\int_0^1 e^{2t}t\,dt=\left|\begin{matrix}u=t & dv=e^{2t}dt\\ du=dt & v=\frac{1}{2}e^{2t}\end{matrix}\right|=2\frac{1}{2}te^{2t}\Big|_0^1-2\frac{1}{2}\int_0^1 e^{2t}dt=$$

$$=e^2-\frac{1}{2}e^{2t}\Big|_0^1=e^2-\frac{1}{2}e^2+\frac{1}{2}e^0=\frac{1}{2}e^2+\frac{1}{2}$$

(#) Dokazati da je vektorsko polje potencijalno i naći njegov potencijal:

$$\vec{v}=2x(y^2+z^2)\vec{i}+2y(x^2+z^2)\vec{j}+2z(x^2+y^2)\vec{k}.$$

Rj. Vektorsko polje $\vec{v}$ je potencijalno ako je $rot\,\vec{v}=\vec{0}$.

Rotor vektorskog polja $rot\,\vec{v}$ se računa:

$$rot\,\vec{v}=\nabla\times\vec{v}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ V_x&V_y&V_z\end{vmatrix}$$

(vektorski proizvod Nabla (∇) operatora i vektorskog polja $\vec{v}$)

$V_x=2x(y^2+z^2)$, $V_y=2y(x^2+z^2)$, $V_z=2z(x^2+y^2)$

$$\frac{\partial V_x}{\partial y}=4xy\qquad \frac{\partial V_y}{\partial x}=4xy\qquad \frac{\partial V_z}{\partial x}=4xz$$

$$\frac{\partial V_x}{\partial z}=4xz\qquad \frac{\partial V_y}{\partial z}=4yz\qquad \frac{\partial V_z}{\partial y}=4yz$$

$$rot\,\vec{v}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ V_x&V_y&V_z\end{vmatrix}=\vec{i}(4yz-4yz)-\vec{j}(4xz-4xz)+\vec{k}(4xy-4xy)$$

$=(0,0,0)=\vec{0}$ vektorsko polje je potencijalno

Potencijal polja $\vec{v}$ je f-ja u za koju vrijedi $\vec{v}=grad\,u$.

$$grad\,u=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)$$

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=2x(y^2+z^2)\\ \frac{\partial u}{\partial y}&=2y(x^2+z^2)\\ \frac{\partial u}{\partial z}&=2z(x^2+y^2)\end{aligned}\right\}\dots(1)$$

$u=u(x,y,z)$

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$$

$$u=\int 2x(y^2+z^2)dx+\varphi(y,z)$$

$u=x^2(y^2+z^2)+\varphi(y,z)$

$$\left.\begin{aligned}\frac{\partial u}{\partial y}&=2yx^2+\varphi'_y\\ \frac{\partial u}{\partial z}&=2x^2z+\varphi'_z\end{aligned}\right\}\dots(2)$$

(1) i (2) $\Rightarrow$ $\varphi'_y=2yz^2$, $\varphi'_z=2zy^2$ ... (*)

Odredimo f-ju φ: $\varphi=\int 2yz^2dy+\psi(z)$

$\varphi=y^2z^2+\psi(z)$

$\varphi'=2y^2z+\psi'$ ... (**)

(*) i (**) $\Rightarrow$ $\psi'_z=0$ $\Rightarrow$ $\psi(z)=c$

$\Rightarrow$ $\varphi=y^2z^2+c$ $\Rightarrow$ $u=x^2y^2+x^2z^2+y^2z^2+c$

Potencijal vektorskog polja je $u=x^2y^2+x^2z^2+y^2z^2+C$

Odrediti brojeve a i b tako da vektorsko polje $\vec{v}=(yz+axy,\ xz+bx^2+yz^2,\ axy+y^2z)$ bude potencijalno i za dobijeno polje izračunati njegovu cirkulaciju duž pravolinijske konture od tačke $A(1,1,1)$ prema tački $B(2,2,2)$.

R: Za vektorsko polje $\vec{v}$ $(\vec{v}=(v_x,v_y,v_z))$ kažemo da je potencijalno ako je $\mathrm{rot}\,\vec{v}=\vec{0}$. Znamo da

$$\mathrm{rot}\,\vec{v}=\nabla\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix}$$

$$\mathrm{rot}\,\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz+axy & xz+bx^2+yz^2 & axy+y^2z\end{vmatrix}=$$

$$=(ax+2yz-x-2yz,\ -(ay-y),\ z+2bx-z-ax)$$

$$=(ax-x,\ y-ay,\ 2bx-ax)$$

$$\mathrm{rot}\,\vec{v}=\vec{0}\Rightarrow\begin{aligned}ax-x&=0\\ y-ay&=0\\ 2bx-ax&=0\end{aligned}\qquad a=1,\quad b=\frac{1}{2}$$

Za $a=1$ i $b=\frac{1}{2}$ vektorsko polje $\vec{v}$ je potencijalno polje.

Cirkulaciju vektorskog polja $\vec{v}$ duž krive c tražimo po formuli

$$C=\int_c \vec{v}\,d\vec{r}=\int_c v_x\,dx+v_y\,dy+v_z\,dz$$

Kriva c je dio prave od tačke $A(1,1,1)$ (x_1,y_1,z_1) do tačke $B(2,2,2)$ (x_2,y_2,z_2).

Kako glasi jednačina prave u prostoru kroz dvije tačke?

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}\Longrightarrow\frac{x-1}{1}=\frac{y-1}{1}=\frac{z-1}{1}\ (=t)\qquad \begin{aligned}x-1&=t\\ y-1&=t\\ z-1&=t\end{aligned}$$

Kriva c u parametarskom obliku

$$c:\begin{cases}x=t+1 & dx=dt\\ y=t+1 & dy=dt\\ z=t+1 & dz=dt\\ 0\le t\le 1\end{cases}$$

U našem slučaju

$$C=\int_c (yz+xy)\,dx+(xz+\tfrac{1}{2}x^2+yz^2)\,dy+(xy+y^2z)\,dz=$$

$$=\int_0^1\left[(t+1)^2+(t+1)^2+(t+1)^2+\frac{1}{2}(t+1)^2+(t+1)^3+(t+1)^2+(t+1)^3\right]dt=$$

$$=\left|\ d(t+1)=dt\ \right|=\int_0^1\left[\frac{9}{2}(t+1)^2+2(t+1)^3\right]d(t+1)=$$

$$=\frac{9}{2}\cdot\frac{(t+1)^3}{3}\Big|_0^1+2\cdot\frac{(t+1)^4}{4}\Big|_0^1=\frac{9}{6}(8-1)+\frac{1}{2}(16-1)$$

$$=\frac{63}{6}+\frac{15\cdot 3}{2\cdot 3}=\frac{108}{6}=18\quad \text{traženo rješenje}$$

Zadaci za vježbu

Vektorsko polje, divergencija i rotor

4401. Naći vektorske linije homogenog polja $A(P) = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ (a, b i c su konstante).

4402. Naći vektorske linije ravnog polja $A(P) = -\omega y\boldsymbol{i} + \omega x\boldsymbol{j}$, ($\omega$ je konstanta).

4403. Naći vektorske linije polja $A(P) = -\omega y\boldsymbol{i} + \omega x\boldsymbol{j} + h\boldsymbol{k}$ (ω i h su konstante).

4404. Naći vektorske linije polja:

1) $A(P) = (y+z)\boldsymbol{i} - x\boldsymbol{j} - x\boldsymbol{k}$;

2) $A(P) = (z-y)\boldsymbol{i} + (x-z)\boldsymbol{j} + (y-x)\boldsymbol{k}$:

3) $A(P) = x(y^2-z^2)\boldsymbol{i} - y(z^2+x^2)\boldsymbol{j} + z(x^2+y^2)\boldsymbol{k}$.

U zadacima 4405 — 4408 izračunati divergenciju i rotor datih vektorskih polja.

4405. $A(P) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

4406. $A(P) = (y^2+z^2)\boldsymbol{i} + (z^2+x^2)\boldsymbol{j} + (x^2+y^2)\boldsymbol{k}$.

4407. $A(P) = x^2yz\boldsymbol{i} + xy^2z\boldsymbol{j} + xyz^2\boldsymbol{k}$.

4408. $A(P) = \text{grad}\,(x^2+y^2+z^2)$.

4409. Sila $F\boldsymbol{i}$ konstantnog intenziteta F obrazuje vektorsko polje; izračunati divergenciju i rotor toga polja.

Rješenja

4401. Prave paralelne vektoru $A\{a, b, c,\}$: $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$.

4402. Krugovi sa centrom u kodrdinatnom početku.

4403. Zavojnice sa visinom hoda $\frac{2\pi h}{\omega}$, koje leže na cilindrima čije se ose poklapaju sa z-osom: $x = R\cos(\omega t + \alpha)$, $y = R\sin(\omega t + \alpha)$, $z = ht + z_0$, pri čemu su R, α i z_0 proizvoljne konstante.

4404. 1) Krugovi $x^2+y^2+z^2 = R^2$, $y - z + C = 0$, po kojima ravni paralelne simetralnoj ravni $y - z = 0$ presecaju sfere sa zajedničkim centrom u koordinatnom početku (R i C su proizvoljne konstante.

2) Krugovi $x^2+y^2+z^2 = R^2$, $x + y + z = C$ po kojima ravni, koje od koordinatnih osa odsecaju odsečke iste dužine i znaka, presecaju sfere sa zajedničkim controm u koordinatnom početku.

3) Krive po kojima se presecaju sfere $x^2+y^2+z^2 = R^2$ i hiperbolični paraboloidi $zy = Cx$.

4405. div $A = 3$, rot $A = 0$.

4406. div $A = 0$, rot $A = 2[(y-z)\boldsymbol{i} + (z-x)\boldsymbol{j} + (x-y)\boldsymbol{k}]$.

4407. div $A = 6xyz$, rot $A = x(z^2-y^2)\boldsymbol{i} + y(x^2-z^2)\boldsymbol{j} + z(y^2-x^2)\boldsymbol{k}$.

4408. div $A = 6$, rot $A = 0$.

4409. div $A = 0$, rot $A = 0$.

4410. Ravno vektorsko polje definisano je silom obrnuto proporcionalnom kvadratu odstojanja njene napadne tačke od koordinatnog početka i usmerenom prema koordinatnom početku (npr. ravno električno polje obrazovano naelektrisanom materijalnom tačkom); naći divergenciju i rotor polja.

4411. Naći divergenciju i rotor prostranog polja ako je sila polja podčinjena istim uslovima kao i u zadatku 4410.

4412. Vektorsko polje je definisano silom obrnuto proporcionalnom odstupanju njene napadne tačke od z-ose, normalnom na tu osu i usmerenom prema njoj; izračunati divergenciju i rotor toga polja.

4413. Vektorsko polje je definisano silom obrnuto proporcionalnom odstojanju njene napadne tačke od ravni xOy i usmerenom prema koordinatnom početku; izračunati divergenciju tog polja.

4414. Izračunati div $(a\boldsymbol{r})$ ako je a konstantan skalar.

4415. Dokazati relaciju

$$\text{div}\,(\varphi A) = \varphi\,\text{div}\,A + (A\,\text{grad}\,\varphi),$$

u kojoj je $\varphi = \varphi(x, y, z)$ skalarna funkcija.

4416. Izračunati div $\boldsymbol{b}(\boldsymbol{r}\cdot\boldsymbol{a})$ i div $\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{a})$ ako su $\boldsymbol{a}$ i $\boldsymbol{b}$ konstantni vektori.

4417. Izračunati div $(\boldsymbol{a}\times\boldsymbol{r})$ ako je $\boldsymbol{r}$ konstantan vektor.

4418. Ne prelazeći na koordinate izračunati divergenciju vektorskog polja:

1) $A(P) = \boldsymbol{r}(\boldsymbol{a}\boldsymbol{r}) - 2\boldsymbol{a}r^2$, 2) $A(P) = \frac{\boldsymbol{r}-\boldsymbol{r}_0}{|\boldsymbol{r}-\boldsymbol{r}_0|^3}$,

3) $\text{grad}\,\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|}$

Rješenja

4410. div $A = \frac{k}{r^2}$, gde je k koeficijent proporcionalnosti, a r—odstojanje napadne tačke sile od koordinatnog početka; rot $A = 0$.

4411. div $A = 0$, rot $A = 0$.

4412. div $A = 0$, rot $A = 0$. U tačkama z-ose polje nije definisano.

4413. div $A = -\frac{k}{z\sqrt{x^2+y^2+z^2}}$, gde je k koeficijent proporcionalnosti. U tačkama ravni Oxy polje nije definisano.

4414. $3a$. **4416.** div $\boldsymbol{b}(\boldsymbol{r}\boldsymbol{a}) = (\boldsymbol{a}\boldsymbol{b})$, div $\boldsymbol{r}(\boldsymbol{r}\boldsymbol{a}) = 4(\boldsymbol{r}\boldsymbol{a})$.

4417. 0. **4418.** 1) 0. 2) 0. 3) 0.

4419. Izračunati divergenciju vektorskog polja

$$A(P)=f(|r|)\frac{r}{|r|}.$$

Dokazati da je divergencija ovog polja jednaka nuli samo onda kad je $f(|r|)=\frac{C}{r^2}$ ako je polje prostorno, i $f(|r|)=\frac{C}{|r|}$ ako je polje ravno, pri čemu je C proizvoljna skalarna konstanta.

4420. Dokazati da je

$$\operatorname{rot}[A_1(P)+A_2(P)]=\operatorname{rot}A_1(P)+\operatorname{rot}A_2(P).$$

4421. Izračunati $\operatorname{rot}[\varphi A(P)]$, ako je $\varphi=\varphi(x, y, z)$ skalarna funkcija.

4422. Izračunati $\operatorname{rot} r\,a$ ako je r intenzitet vektora položaja tačke, a a je konstantan vektor.

4423. Izračunati $\operatorname{rot}(a\times r)$ ako je a konstantan vektor.

4424. Kruto telo obrće se konstantnom ugaonom brzinom ω oko ose: naći divergenciju i rotor polja linearnih brzina.

4425. Dokazati relaciju

$$n(\operatorname{grad}(A\,n)-\operatorname{rot}(A\times n))=\operatorname{div}A,$$

ako je n jedinični konstantan vektor.

Diferencijalne operacije vektorske analize (grad, div, rot) zgodno je obeležavati pomoću simboličnog vektora ∇ (Hamiltonov „nabla" operator):

$$\nabla=\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k.$$

Primenu ovog operatora na ovu ili onu (skalarnu ili vektorsku veličinu) treba shvatiti ovako: po pravilima vektorske algebre treba pomnožiti vektor ∇ datom veličinom, a zatim množenje simbola $\frac{\partial}{\partial x}$ i tsl. veličinom S shvatiti kao izračunavanje odgovarajućeg izvoda. Tada je $\operatorname{grad}u=\nabla u$; $\operatorname{div}A=\nabla A$; $\operatorname{rot}A=\nabla\times A$.

Pomoću Hamiltonova operatora mogu se predstaviti i diferencijalne operacije drugog reda: $\operatorname{div}\operatorname{grad}u=\nabla\nabla u$; $\operatorname{rot}\operatorname{grad}u=\nabla\times\nabla u$; $\operatorname{grad}\operatorname{div}A=\nabla(\nabla A)$; $\operatorname{div}\operatorname{rot}A=\nabla(\nabla\times A)$; $\operatorname{rot}\operatorname{rot}A=\nabla\times(\nabla\times A)$.

4426. Dokazati da je $r\cdot\nabla r^n=n\,r^n$, pri čemu je r vektor položaja tačke.

4427. Dokazati relacije:

1) $\operatorname{rot}\operatorname{grad}u=0$; 2) $\operatorname{div}\operatorname{rot}A=0$.

4428. Dokazati da je

$$\operatorname{div}\operatorname{grad}u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}.$$

(Ovaj se izraz naziva Laplasovim operatorom i obično se obeležava sa Δu. Pomoću Hamiltonova operatora ova se veličina može pisati u obliku $\Delta u=(\nabla\nabla)u=\nabla^2 u$).

4429. Dokazati da je

$$\operatorname{rot}\operatorname{rot}A(P)=\operatorname{grad}\operatorname{div}A(P)-\Delta A(P),$$

pri čemu je

$$\Delta A(P)=\Delta A_x i+\Delta A_y j+\Delta A_z k.$$

Potencijal

4430. Vektorsko polje definisano je konstantnim vektorom A; uveriti se da to polje ima potencijal i naći taj potencijal.

4431. Vektorsko polje definisano je silom proporcionalnom odstojanju napadne tačke od koordinatnog početka i usmerenom prema koordinatnom početku; pokazati da je to polje konzervativno i naći njegov potencijal.

4432. Sile polja su obrnuto proporcionalne odstojanju njihovih napadnih tačaka od ravni Oxy i usmerene su prema koordinatnom početku; hoće li polje biti konzervativno?

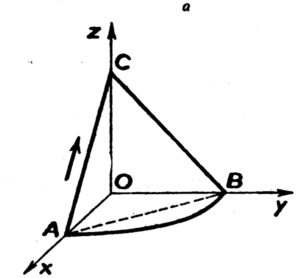
4433. Sile polja su obrnuto proporcionalne kvadratu odstojanja njihovih napadnih tačaka od z-ose i usmerene prema koordinatnom početku; hoće li polje biti konzervativno?

4434. Vektorsko polje definisano je silom obrnuto proporcionalnom odstojanju njene napadne tačke od z-ose, normalnom na tu osu i usmerenom ka njoj; pokazati da je to polje konzervativno i naći njegov potencijal.

4435. Linearne brzine tačaka krutog tela koje se obrće oko neke ose obrazuju vektorsko polje; je li to polje potencijalno?

4436. Sile polja definisane su ovako: $A(P)=f(r)\frac{r}{r}$ (tzv. centralno polje; $r=\sqrt{x^2+y^2+z^2}$); pokazati da je potencijal polja: $u(x, y, z)=\int_a^r f(r)\,dr$ i odavde kao specijalan slučaj izvesti potencijal polja sila privlačenja koje potiču od tačkaste mase, i potencijal polja u zadatku 4431.

4437. Naći rad sila polja $A(p)=xy\,i+yz\,j+xz\,k$ pri pomeranju tačke po zatvorenoj krivoj koja se sastoji iz odsečka prave $x+z=1$, $y=0$, četvrtine kružne linije $x^2+y^2=1$, $z=0$, i odsečka prave $y+z=1$, $x=0$ (sl. 78), — u smeru naznačenom na slici. Koliki će biti taj rad ako se luk BA zameni izlomljenom linijom BOA ili pravolinijskim odsečkom BA?



Sl. 78

Rješenja

4419. $\operatorname{div}A=\frac{2f(r)}{r}+f'(r)$, ako je polje prostorno, i $\operatorname{div}A=\frac{f(r)}{r}+f'(r)$ ako je polje ravno.

4421. $\varphi\operatorname{rot}A+(\operatorname{grad}\varphi\times A)$. **4422.** $\frac{r\times A}{r}$.

4423. $2a$. **4424.** $\omega\,n_0$, gde je n_0 jedinični vektor paralelan osi obrtanja.

4430. $u=A\,r+C$. **4431.** $u=-\frac{1}{2}k(x^2+y^2+z^2)+C$. **4432.** Neće. **4433.** Neće.

4434. $u=-\frac{1}{2}\ln(x^2+y^2)+C$. **4435.** Nema.

4437. $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{2}$. **4438.** $k\delta\ln\frac{\sqrt{(l-x)^2+y^2}+l-x}{\sqrt{(l+x)^2+y^2}-l-x}$.

Cirkulacija i fluks vektorskog polja

Neka je $\vec{v}=(V_x, V_y, V_z)$ dato vektorsko polje.
Cirkulacija vektorskog polja $\vec{v}$ duž krive c je integral

$$C=\int_c \vec{v}\, d\vec{r}=\int_c V_x\,dx+V_y\,dy+V_z\,dz \qquad \text{gdje je } \vec{r}=(x,y,z),\ d\vec{r}=(dx,dy,dz)$$

Ako je c zatvorena kontura možemo koristiti formulu Stoksa u vektorskom obliku

$$C=\oint_c \vec{v}\,d\vec{r}=\iint_S \vec{n}\cdot \operatorname{rot}\vec{v}\,dS=\iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} dS$$

Fluks (tok, proticanje) vektorskog polja (kroz površ S) je površinski integral

$$\Phi=\iint_S \vec{v}\cdot\vec{n}\,dS=\iint_S (V_x\cos\alpha+V_y\cos\beta+V_z\cos\gamma)\,dS$$

$$=\iint_S V_x\,dy\,dz+V_y\,dx\,dz+V_z\,dx\,dy$$

Ako je S zatvorena površ, fluks polja se može računati pomoću formule Gauss-Ostrogradski

$$\Phi=\iint_S \vec{v}\cdot\vec{n}\,dS=\iiint_\Omega \operatorname{div}\vec{v}\,dx\,dy\,dz=\iiint_\Omega \left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right)dx\,dy\,dz$$

gdje je Ω oblast u prostoru koja je ograničena površinom S.

(#) Izračunati cirkulaciju polja $\vec{v}=x\vec{i}+y\vec{j}+(x+y-1)\vec{k}$ duž odsječka prave između tačaka $A(1,1,1)$ i $B(2,3,4)$.

Rj. Cirkulacija vektorskog polja $\vec{v}=(V_x,V_y,V_z)$ duž krive c je integral

$$C=\int_c V_x\,dx+V_y\,dy+V_z\,dz.$$

U našem slučaju $\vec{v}=(x, y, x+y-1)$, dok je c dio prave između tačaka $A(1,1,1)$ i $B(2,3,4)$.
Imamo krivolinijski integral druge vrste

$$C=\int_c x\,dx+y\,dy+(x+y-1)\,dz$$

$A(1,1,1)$
$B(2,3,4)$

Kako glasi jednačina prave kroz dvije tačke u prostoru?

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

$$\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3} \quad (=t)$$

Napišimo pravu u parametarskom obliku:

$x=t+1$
$y=2t+1$
$z=3t+1$

Dio prave između tačke $A(1,1,1)$ i $B(2,3,4)$ je za $t\in[0,1]$.

$dx=dt,\ dy=2\,dt,\ dz=3\,dt$

$$C=\int_0^1 (t+1)\,dt+(2t+1)\,2\,dt+(3t+1)\,3\,dt=\int_0^1 (t+1+4t+2+9t+3)\,dt$$

$$=\int_0^1 (14t+6)\,dt=14\cdot\frac{1}{2}t^2\Big|_0^1+6t\Big|_0^1=7+6=13$$ vrijednost cirkulacije polja

(#) Izračunati tok (fluks) vektora $\vec{v} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ kroz sferu $x^2+y^2+z^2=R^2$.

Rj. $\vec{v} = (v_x, v_y, v_z) = (x^3, y^3, z^3)$

Tok vektorskog polja (kroz površ S) je površinski integral

$$\phi = \iint_S v_x\,dy\,dz + v_y\,dx\,dz + v_z\,dx\,dy$$

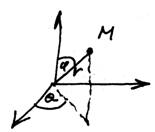
Kako je data zatvorena površina S to možemo upotrebiti formulu Gauss-Ostrogradski

$$\iint_S v_x\,dy\,dz + v_y\,dx\,dz + v_z\,dx\,dy = \iiint_\Omega \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)dx\,dy\,dz$$

$\frac{\partial v_x}{\partial x} = 3x^2$, $\frac{\partial v_y}{\partial y} = 3y^2$, $\frac{\partial v_z}{\partial z} = 3z^2$, Ω oblast ograničena sferom $x^2+y^2+z^2=R^2$

$$\phi = \iiint_\Omega 3(x^2+y^2+z^2)\,dx\,dy\,dz \overset{(\Delta)}{=}$$

uvedimo sferne koordinate



$$\Omega' : \begin{cases} 0 \le r \le R \\ 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{cases}$$

$x = r\sin\varphi\cos\theta$
$y = r\sin\varphi\sin\theta$
$z = r\cos\varphi$
$dx\,dy\,dz = r^2\sin\varphi$
$x^2+y^2+z^2 = r^2[\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi] = r^2$

$$\overset{(\Delta)}{=} 3\iiint_{\Omega'} r^2\, r^2 \sin\varphi\,dr\,d\varphi\,d\theta =$$

$$= 3\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\,d\varphi \int_0^R r^4\,dr = 3\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\,\frac{1}{5}r^5\Big|_0^R d\varphi = 3\,\frac{R^5}{5}\int_0^{2\pi} (-\cos\varphi)\Big|_0^{\pi} d\theta$$

$$= \frac{6R^5}{5}\pi\,\theta\Big|_0^{2\pi} = \frac{12R^5}{5}\pi \quad \text{traženi tok vektora kroz sferu}$$

(#) Izračunati cirkulaciju vektorskog polja $\vec{v} = -y\vec{i} + x\vec{j} + a\vec{k}$ (a = konstanta) duž kruga $(x-2)^2+y^2=1$, $z=0$.

Rj. $\vec{v} = -y\vec{i} + x\vec{j} + a\vec{k}$

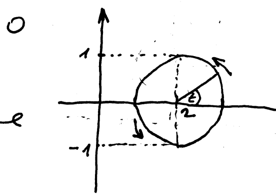
$c: (x-2)^2 + y^2 = 1,\ z = 0$

$$C = \int_c \vec{v}\,d\vec{r} = \int_c v_x\,dx + v_y\,dy + v_z\,dz$$

cirkulacija polja $\vec{v}$

Imamo krivolinijski integral

$$C = \int_c -y\,dx + x\,dy + a\,dz \quad \text{gdje je } c: \begin{cases} (x-2)^2+y^2=1 \\ z=0 \end{cases}$$



Parametrizirajmo kružnicu tj. uvedimo smjene

$$\begin{cases} x-2 = \cos t \\ y = \sin t \\ z = 0 \end{cases} \quad 0 \le t \le 2\pi \qquad dx = -\sin t\,dt,\ dy = \cos t\,dt,\ dz = 0 \qquad x = 2 + \cos t$$

$$C = \int_0^{2\pi} (-\sin t)(-\sin t)\,dt + (2+\cos t)\cos t\,dt + 0 =$$

$$\int_0^{2\pi} (\sin^2 t + 2\cos t + \cos^2 t)\,dt = \int_0^{2\pi} (1 + 2\cos t)\,dt = (t + 2\sin t)\Big|_0^{2\pi} = 2\pi$$

II način: pomoću Stoksove formule

$$C = \int_c \vec{v}\,d\vec{r} = \iint_S \vec{n}\cdot \operatorname{rot}\vec{v}\,dS = \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} dS$$

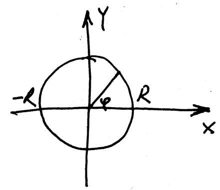
$$\operatorname{rot}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & a \end{vmatrix} = 0\vec{i} + 0\vec{j} + 2\vec{k} = (0,0,2)$$

$$C = \iint_S \vec{n}\cdot\operatorname{rot}\vec{v}\,dS = \iint_S 2\cos\gamma\,dS = 2\underbrace{\iint_S dx\,dy}_{\text{površina kruga}} = 2\cdot 1^2\cdot\pi = 2\pi$$

| Iz formule Stoksa znamo da je $\cos\gamma\,dS = dx\,dy$ |

Izračunati cirkulaciju vektorskog polja $\vec{v} = x^2y^3\vec{i} + \vec{j} + z\vec{k}$ duž kružnice c koja je data kao presjek kružnice $x^2+y^2=R^2$ i xOy ravni.

Rj. $c: \begin{cases} x^2+y^2=R^2 \\ z=0 \end{cases}$



$C = \int_c \vec{v}\, d\vec{r} = \int_c V_x\,dx + V_y\,dy + V_z\,dz$

cirkulacija polja $\vec{v}$

I način

Parametrizirajmo kružnicu $\begin{cases} x=R\cos t \\ y=R\sin t \\ z=0 \end{cases}$ ... ZAVRŠITI ZA VJEŽBU

II način Pomoću formule Stoksa:

$$C = \int_c \vec{v}\, d\vec{r} = \iint_S \vec{n}\cdot \operatorname{rot}\vec{v}\, dS \qquad \operatorname{rot}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = (0,0,-3x^2y^2)$$

$$C = \iint_S (\cos\alpha, \cos\beta, \cos\gamma)\cdot(0,0,-3x^2y^2)\, dS = \iint_S -3x^2y^2\cos\gamma\, dS =$$

$$= -3\iint_S x^2y^2\,dx\,dy \quad \text{gdje je sad } S: \{x^2+y^2=R^2$$

Uvodimo polarne koordinate $x = r\cos\varphi$, $y = r\sin\varphi$ $\Rightarrow$ $S': \begin{cases} 0\le r\le R \\ 0\le\varphi\le 2\pi \end{cases}$ $dx\,dy = r\,dr\,d\varphi$

$$C = -3\iint_{S'} r^2\cos^2\varphi\, r^2\sin^2\varphi\cdot r\,dr\,d\varphi = -3\int_0^R r^5\left[\int_0^{2\pi}\frac{1}{4}\cdot\underbrace{4\cos^2\varphi\sin^2\varphi}_{(\sin 2\varphi)^2}\,d\varphi\right]dr$$

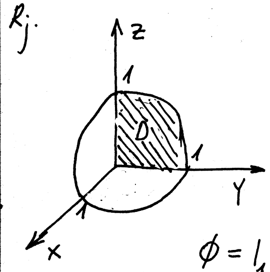
$$= -3\int_0^R r^5\left[\int_0^{2\pi}\frac{1}{4}\sin^2 2\varphi\, d\varphi\right]dr = -\frac{3}{4}\int_0^R r^5\left[\int_0^{2\pi}\frac{1-\cos 4\varphi}{2}\,d\varphi\right]dr$$

$$= -\frac{3}{4}\left[\frac{1}{2}\varphi\Big|_0^{2\pi} - \frac{1}{4}\sin 4\varphi\Big|_0^{2\pi}\right]\cdot\frac{1}{6}r^6\Big|_0^R = -\frac{1}{8}R^6\cdot\pi$$

$1 = \cos^2 2\varphi + \sin^2 2\varphi$
$\cos 4\varphi = \cos^2 2\varphi - \sin^2 2\varphi$

Naći fluks polja $\vec{v} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ kroz dio sfere $x^2+y^2+z^2=1$ u I oktantu.

Rj.



I način

$$\Phi = \iint_S \vec{v}\cdot\vec{n}\, dS = \iint_S (V_x\cos\alpha + V_y\cos\beta + V_z\cos\gamma)\, dS$$
$$= \iint_S V_x\,dy\,dz + V_y\,dx\,dz + V_z\,dx\,dy$$

$$\Phi = I_1 + I_2 + I_3 = \iint_S xy\,dy\,dz + \iint_S yz\,dx\,dz + \iint_S zx\,dx\,dy$$

Zbog simetrije $I_1 = I_2 = I_3$ pa je $\Phi = 3I_1$. Računamo samo I_1

$$I_1 = \iint_S xy\,dy\,dz = +\iint_D \sqrt{1-(y^2+z^2)}\,y\,dy\,dz \quad \text{gdje je } D: y^2+z^2\le 1,\ x\ge 0,\ y\ge 0$$

$x^2 = 1-(y^2+z^2)$
$x = \pm\sqrt{1-(y^2+z^2)}$

Vektor normale zaklapa ugao $\alpha\in(0,\frac{\pi}{2})$ sa x-osom. $\cos\alpha > 0$ (u I oktantu).

uzimamo + jer smo u prvom oktantu

Uvodimo polarne koordinate $y = r\cos\varphi$, $z = r\sin\varphi$

$D': \begin{cases} 0\le\varphi\le\frac{\pi}{2} \\ 0\le r\le 1 \\ dy\,dz = r\,d\varphi\,dr \end{cases}$ $\quad y^2+z^2 = r^2$

$$I_1 = \iint_{D'} r\cos\varphi\sqrt{1-r^2}\cdot r\,d\varphi\,dr = \int_0^1 r^2\sqrt{1-r^2}\left[\int_0^{\frac{\pi}{2}}\cos\varphi\,d\varphi\right]dr = \int_0^1 r^2\sqrt{1-r^2}\cdot 1\,dr$$

$$= \begin{vmatrix} r = \sin t \\ dr = \cos t\,dt \end{vmatrix} = \int_0^1 \sin^2 t\sqrt{1-\sin^2 t}\,\cos t\,dt = \int_0^1 \sin^2 t\cos^2 t\,dt = \ldots = \frac{3\pi}{16}$$

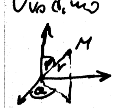
↳ u prethodnom zadatku smo imali slično

II način: Kako je S zatvorena površ možemo primjeniti formulu Gauss-Ostrogradski.

$$\Phi = \iint_S \vec{v}\cdot\vec{n}\,dS = \iiint_\Omega \operatorname{div}\vec{v}\,dx\,dy\,dz = \iiint_\Omega\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)dx\,dy\,dz$$

U našem slučaju $\Phi = \iiint_\Omega (x+y+z)\,dx\,dy\,dz$ gdje je $\Omega: \begin{cases} x^2+y^2+z^2\le 1 \\ x\ge 0,\ y\ge 0 \\ z\ge 0 \end{cases}$

Uvodimo sferne koordinate



$x = r\sin\varphi\cos\theta$
$y = r\sin\varphi\sin\theta$
$z = r\cos\varphi$

$\Rightarrow \Omega': \begin{cases} 0\le r\le 1 \\ 0\le\varphi\le\frac{\pi}{2} \\ 0\le\theta\le\frac{\pi}{2} \end{cases}$ $\quad dx\,dy\,dz = r^2\sin\varphi\,d\varphi\,dr\,d\theta$

$$\Phi = \iiint_{\Omega'} (r\sin\varphi\cos\theta + \ldots)\, r^2\sin\varphi \ldots = \ldots = \frac{3\pi}{16}$$

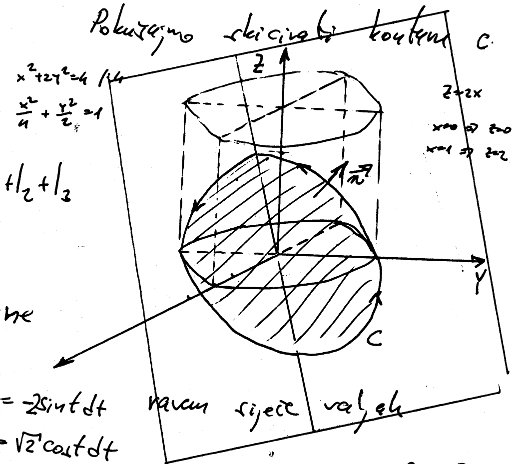
(#) Izračunati cirkulaciju vektorskog polja $\vec{v}=(1, xy^2, yz^2)$ duž konture $x^2+2y^2=4$, $z=2x$.

R.: Cirkulacija vektorskog polja $\vec{v}$ $(\vec{v}=(v_x, v_y, v_z))$ duž krive c je integral

$$C=\int_c \vec{v}\, d\vec{r}=\int_c v_x\,dx+v_y\,dy+v_z\,dz$$

U našem slučaju

$$C=\int_c dx+xy^2\,dy+yz^2\,dz=I_1+I_2+I_3$$



parametriziramo konturu c

kako je $\left(\frac{x}{2}\right)^2+\left(\frac{y}{\sqrt{2}}\right)^2=1$ uvedimo smjene

$$\left.\begin{aligned}\frac{x}{2}&=\cos t\\ \frac{y}{\sqrt{2}}&=\sin t\\ z&=z\end{aligned}\right\} \Rightarrow \begin{aligned}x&=2\cos t\\ y&=\sqrt{2}\sin t\\ z&=4\cos t\\ 0&\le t\le 2\pi\end{aligned} \qquad \begin{aligned}dx&=-2\sin t\,dt\\ dy&=\sqrt{2}\cos t\,dt\\ dz&=-4\sin t\,dt\end{aligned}$$

$\sin^2x+\cos^2x=1$
$\cos^2x-\sin^2x=\cos 2x$
$2\sin^2x=1-\cos 2x$
$2\cos^2x=1+\cos 2x$

$$C=\int_0^{2\pi}\left(-2\sin t+2\cos t\cdot 2\sin^2 t\cdot\sqrt{2}\cos t+\sqrt{2}\sin t\cdot 16\cdot\cos^2 t(-4\sin t)\right)dt$$

pojednostavimo računanje ovog integrala

$$I_1=\int_c dx=\int_0^{2\pi}-2\sin t\,dt=2\cos t\Big|_0^{2\pi}=2(1-1)=0$$

$4\cos^2t\sin^2t=(2\cos t\sin t)^2$

$$I_2=\int_c xy^2\,dy=\int_0^{2\pi}2\cos t\cdot 2\sin^2 t\cdot\sqrt{2}\cos t\,dt=4\sqrt{2}\int_0^{2\pi}\cos^2 t\sin^2 t\,dt=\sqrt{2}\int_0^{2\pi}\sin^2 2t\,dt$$

$$=\frac{\sqrt{2}}{2}\int_0^{2\pi}(1-\cos 2t)\,dt=\frac{\sqrt{2}}{2}\left(t\Big|_0^{2\pi}-\frac{1}{2}\sin t\Big|_0^{2\pi}\right)=\frac{\sqrt{2}}{2}(2\pi-0)=\pi\sqrt{2}$$

$$I_3=\int_c yz^2\,dz=\int_0^{2\pi}\sqrt{2}\sin t\cdot 16\cos^2 t\cdot(-4)\sin t\,dt=\overset{(-16)\cdot 4}{-64\sqrt{2}}\int_0^{2\pi}\sin^2 t\cos^2 t=-16\,I_2=-16\pi\sqrt{2}$$

$$C=\pi\sqrt{2}-16\pi\sqrt{2}=-15\pi\sqrt{2}$$

II način

pomoću Stokesove formule (površinski integral ↓)

$$C=\int_c \vec{v}\,d\vec{r}=\iint_S \vec{n}\,\mathrm{rot}\,\vec{v}\,dS=\iint_S\begin{vmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix}dS$$

gdje je S površ koju zatvara kontura c, $\vec{n}=(\cos\alpha,\cos\beta,\cos\gamma)$ jedinični vektor normale na S

$$\mathrm{rot}\,\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 1 & xy^2 & yz^2\end{vmatrix}=(z^2-0)\vec{i}-(0-0)\vec{j}+(y^2-0)\vec{k}=(z^2,0,y^2)$$

$$C=\iint_S(z^2\cos\alpha+y^2\cos\gamma)\,dS$$

posmatramo ravan $z=2x$ i vektor normale na ovu ravan, zato što je naša elipsa unutar ove ravni

$2x-z=0$ $\quad\vec{n}=(2,0,-1)$

$|\vec{n}|=\sqrt{4+1}=\sqrt{5}$ $\quad\vec{n}_0=\left(\frac{2}{\sqrt{5}},0,-\frac{1}{\sqrt{5}}\right)$

projekcija površi S na xOy ravan je elipsa $x^2+2y^2=4$

prema tome

$$C=\iint_S(z^2\cos\alpha+y^2\cos\gamma)\,dS=\iint_{D'}z^2\,dy\,dz-\iint_{D''}y^2\,dx\,dy$$

$x^2+2y^2=4$
$z=2x$
$\left(\frac{z}{2}\right)^2+2y^2=4$ /·4
$z^2+8y^2=16$ /:16
$\frac{z^2}{16}+\frac{y^2}{2}=1$

Projekcija površi S na yOz ravan je elipsa $D':\left\{\frac{z^2}{16}+\frac{y^2}{2}=1\right.$

$$\iint_{D'}z^2\,dy\,dz=\left|\begin{aligned}\frac{z}{4}&=r\cos\varphi & z&=4r\cos\varphi & dy\,dz&=4\sqrt{2}\,r\,dr\,d\varphi\\ \frac{y}{\sqrt{2}}&=r\sin\varphi & y&=\sqrt{2}\,r\sin\varphi\\ && 0&\le\varphi\le 2\pi\\ && 0&\le r\le 1\end{aligned}\right|=\iint_{D_1'}16r^2\cos^2\varphi\,4\sqrt{2}\,r\,dr\,d\varphi=$$

$$=64\sqrt{2}\int_0^1 r^3\,dr\int_0^{2\pi}\cos^2\varphi\,d\varphi=64\sqrt{2}\int_0^1 r^3\,dr\int_0^{2\pi}\frac{1}{2}(1+\cos 2\varphi)\,d\varphi=32\sqrt{2}\cdot\frac{r^4}{4}\Big|_0^1\left(2\pi-\frac{1}{2}\sin 2\varphi\Big|_0^{2\pi}\right)=16\sqrt{2}\pi$$

Projekcija površi S na xOy ravan je elipsa $D'':\left\{\frac{x^2}{4}+\frac{y^2}{2}=1\right.$

$$\iint_{D''}y^2\,dx\,dy=\left|\begin{aligned}\frac{x}{2}&=r\cos\varphi & x&=2r\cos\varphi & dx\,dy&=2\sqrt{2}\,r\,dr\,d\varphi\\ \frac{y}{\sqrt{2}}&=r\sin\varphi & y&=\sqrt{2}\,r\sin\varphi & 0&\le\varphi\le 2\pi\\ &&&& 0&\le r\le 1\end{aligned}\right|=\int_0^1\int_0^{2\pi}2r^2\sin^2\varphi\cdot 2\sqrt{2}\,r\,dr\,d\varphi=$$

$$=4\sqrt{2}\int_0^1 r^3\,dr\int_0^{2\pi}\sin^2\varphi\,d\varphi=\ldots=\pi\sqrt{2}$$

$$C=16\pi\sqrt{2}-\pi\sqrt{2}=15\pi\sqrt{2}$$

(#) Izračunati cirkulaciju vektorskog polja $\vec{v}=(e^{y-z}, e^{z-x}, e^{x-y})$ duž odsječka prave od tačke $O(0,0,0)$ do tačke $T(1,3,5)$.

Rj: Cirkulacija vektorskog polja $\vec{v}$ duž krive c je integral

$$C=\int_c \vec{v}\,d\vec{r}=\int_c v_x\,dx+v_y\,dy+v_z\,dz \quad \text{gdje je } \vec{r}=(x,y,z),\ d\vec{r}=(dx,dy,dz).$$

$O(0,0,0)$, $T(1,3,5)$. Jednačina prave kroz tačku OT je

$$\frac{x}{1}=\frac{y}{3}=\frac{z}{5}\ (=t)$$

$$\overline{OT}:\begin{cases} x=t\\ y=3t\\ z=5t\\ 0\le t\le 1\end{cases}$$ u parametarskom obliku

$$\int_c e^{y-z}dx+e^{z-x}dy+e^{x-y}dz=\left|\begin{matrix} x=t, & dx=dt & y-z=-2t\\ y=3t, & dy=3\,dt & z-x=4t\\ z=5t, & dz=5\,dt & x-y=-2t\end{matrix}\right|=$$

$$=\int_0^1\left(e^{-2t}+e^{4t}\cdot 3+e^{-2t}\cdot 5\right)dt=\int_0^1\left(6e^{-2t}+3e^{4t}\right)dt$$

$$=\left|\begin{matrix} d(-2t)=-2\,dt\\ dt=-\frac{1}{2}d(-2t)\\ d(4t)=4\,dt\\ dt=\frac{1}{4}d(4t)\end{matrix}\right|=6\cdot\left(-\frac{1}{2}\right)\int_0^1 e^{-2t}d(-2t)+3\cdot\frac{1}{4}\int_0^1 e^{4t}d(4t)=$$

$$=-3e^{-2t}\Big|_0^1+\frac{3}{4}e^{4t}\Big|_0^1=-3\left(e^{-2}-1\right)+\frac{3}{4}\left(e^{4}-1\right)$$

$$=(-3)e^{-2}+\frac{3}{4}e^{4}+3-\frac{3}{4}=(-3)e^{-2}+\frac{3}{4}e^{4}+\frac{9}{4}$$ traženo rješenje

Zadaci za vježbu

Protok (fluks) i cirkulacija (u ravni)

4450. Izračunati protok i cirkulaciju konstantnog vektora A duž proizvoljne zatvorene krive L.

4451. Izračunati protok i cirkulaciju vektora $A(P)=a\mathbf{r}$, pri čemu je a — konstantan skalar, a $\mathbf{r}$ — vektor položaja tačke P, — duž proizvoljne zatvorene krive L.

4452. Izračunati protok i cirkulaciju vektora $A(P)=x\mathbf{i}-y\mathbf{j}$ duž proizvoljne zatvorene krive L.

4453. Izračunati protok i cirkulaciju vektora $A(P)=(x^3-y)\mathbf{i}+(y^3+x)\mathbf{j}$ duž kružnice poluprečnika R sa centrom u koordinatnom početku.

4454. Potencijal polja brzinâ čestica tečnosti je $u=\ln r\ (r=\sqrt{x^2+y^2})$; odrediti količinu tečnosti koja ističe u jedinici vremena kroz zatvorenu konturu opisanu oko kordinatnog početka (protok), i količinu tečnosti koja protiče u jedinici vremena duž te konture (cirkulacija). Koliki će biti rezultat ako centar leži van konture?

4455. Potencijal polja brzinâ čestica tečnosti je $u=\varphi$, pri čemu je $\varphi=\operatorname{arctg}\frac{y}{x}$; odrediti protok i cirkulaciju vektora brzina duž zatvorene konture L.

4456. Potencijal polja brzinâ čestica tečnosti je $u(x,y)=x(x^2-3y^2)$; izračunati količinu tečnosti koja protekne u jedinici vremena kroz pravolinijski odsečak koji spaja koordinatni početak sa tačkom (1,1).

Rješenja

4450. I protok i cirkulacija su jednaki nuli.

4451. Vrednost protoka je $2aS$, gde je S površina oblasti ograničene konturom L; cirkulacija je jednaka nuli.

4452. I protok i cirkulacija su jednaki nuli.

4453. Vrednost protoka je $\frac{2}{3}\pi R^4$, a cirkulacija je $2\pi R^2$.

4454. U slučaju kad koordinatni početak leži unutar konture protok ima vrednost 2π, u protivnom slučaju njegova je vrednost nula; cirkulacija je u oba slučaja jednaka nuli.

4455. Ako koordinatni početak leži unutar konture cirkulacija je 2π, a ako leži van konture vrednost cirkulacije je 0; protok je u oba slučaja jednak nuli.

Zadaci za vježbu

Protok i cirkulacija (u prostoru)

4457. Dokazati da je početak vektora položaja $\boldsymbol{r}$ kroz svaku zatvorenu površinu jednak trostrukoj zapremini tela ograničenog tom površinom.

4458. Izračunati protok vektora položaja kroz bočnu površinu kružnog cilindra (poluprečnik osnove je R, visina H), ako osa cilindra prolazi kroz koordinatni početak.

4459. Koristeći rezultate zadataka 4457 i 4458 utvrditi koliki je protok vektora položaja kroz obe osnove cilindra prethodnog zadatka.

4460. Izračunati protok vektora položaja kroz bočnu površinu kružnog konusa čija osnova leži u ravni xOy, a osa mu se poklapa sa z-osom. (Visina konusa je $=1$, a poluprečnik osnove je $=2$).

4461. Naći protok vektora $A(P) = xy\,\boldsymbol{i} + yz\,\boldsymbol{j} + xz\,\boldsymbol{k}$ kroz onaj deo površine sfere $x^2 + y^2 + z^2 = 1$ koji leži u prvom oktantu.

4462*. Naći protok vektora $A(P) = yz\,\boldsymbol{i} + xz\,\boldsymbol{j} + xy\,\boldsymbol{k}$ kroz bočnu površinu piramide sa vrhom u tački $S(0, 0, 2)$, čija je osnova trougao sa temenima $0(0, 0, 0)$, $A(2, 0, 0)$ i $B(0, 1, 0)$.

4463. Izračunati cirkulaciju vektora položaja jednog zavoja AB zavojnice $x = a\cos t$, $y = a\sin t$, $z = bt$, ako su A i B tačke koje odgovaraju vrednostima 0 i 2π parametra t.

4464. Kruto telo se obrće konstantnom ugaonom brzinom ω oko z-ose; izračunati cirkulaciju polja linearnih brzina duž kružne linije poluprečnika R, čiji centar leži na osi obrtanja a ravan joj je normalna na tu osu, — u smeru u kom se vrši obrtanje.

4465*. Izračunati protok rotora vektorskog polja $A(P) = y\,\boldsymbol{i} + z\,\boldsymbol{j} + x\,\boldsymbol{k}$ kroz površinu obrtnog paraboloida $z = 2(1 - x^2 - y^2)$ koju od njega odseca ravan $z = 0$

Rješenja

4456. 2. **4458.** $2\pi R^2 H$. **4459.** $\pi R^2 H$.

4460. 4π. Izračunati protok kroz osnovu konusa i iskoristiti rezultat zadatka 4457.

4461 $\dfrac{3\pi}{16}$.

4462*. $\dfrac{1}{6}$. Primeniti formulu Ostrogradskog i izračunati protok kroz osnovu piramide

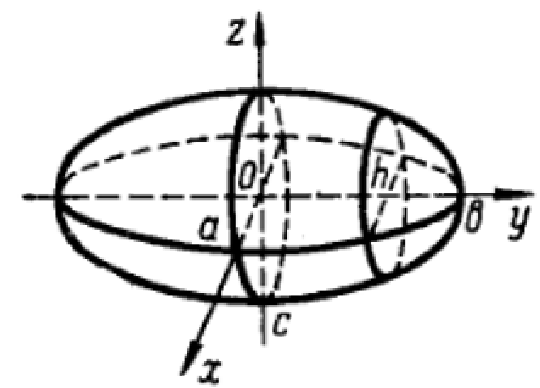
4463. $2\pi^2 b^2$. **4464.** $2\pi\omega R^2$.

4465. $-\pi$. Primeniti Škotsovu formulu uzimajući za konturu L krivu po kojoj ravan Oxy preseca paraboloid.

---

1. Naći protok (fluks) vektorskog polja $\vec{p} = x\vec{i} - y^2\vec{j} + \left(x^2 + y^2 - 1\right)\vec{k}$ kroz elipsoidu $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

Rješenje:



Slika 1: elipsoid

Kako je S zatvorena površ možemo primijeniti formulu Gauss - Ostrogradski

$$\Phi = \iint\limits_S \vec{p}\vec{n}\, ds = \iiint\limits_\Omega div\,\vec{p}\, dxdydz = \iiint\limits_\Omega \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dxdydz$$

Imamo:

$$\vec{p} = \left(v_x, v_y, v_z\right) = \left(x, -y^2, x^2 + y^2 - 1\right)$$

$$\frac{\partial v_x}{\partial x} = 1,\ \frac{\partial v_y}{\partial y} = -2y,\ \frac{\partial v_z}{\partial z} = 0$$

Oblast Ω je ograničena elipsoidom (vidi sliku 1) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

$$\Phi = \iiint\limits_\Omega (1 - 2y)\, dxdydz = (*)$$

Uvedimo sferne koordinate:

$$x = ra\,sin\varphi\,cos\theta$$
$$y = rb\,sin\varphi\,sin\theta$$
$$z = rc\,cos\varphi$$
$$dx\,dy\,dz = r^2 sin\varphi abc\,dr\,d\varphi\,d\theta$$
$$\Omega' = \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{cases}$$

$$(*) = \iiint_{\Omega'} (1 - 2rb\ \sin\varphi\ \sin\theta)\ r^2 \sin\varphi abc\ dr\ d\varphi\ d\theta$$
$$= abc\int_0^1 r^2\ dr\int_0^{\pi} \sin\varphi\ d\varphi \int_0^{2\pi} \left(1 - 2rb\ \sin\varphi\ \sin\theta\right)\ d\theta$$
$$= abc\int_0^1 r^2\ dr\int_0^{\pi} (2\pi - 2rb\ \sin\varphi(-\cos 2\pi + \cos 0)\sin\varphi\ d\varphi$$
$$= abc\int_0^1 r^2\ dr\int_0^{\pi} (2\pi - 2rb\ \sin\varphi\left(-1+1\right))\sin\varphi\ d\varphi$$
$$= abc\int_0^1 r^2\ dr\int_0^{\pi} 2\pi\ \sin\varphi\ d\varphi$$
$$= 2\pi abc\int_0^1 r^2\ dr\int_0^{\pi} \sin\varphi\ d\varphi$$
$$= 2\pi abc\int_0^1 (-\cos\pi + \cos 0) r^2\ dr$$
$$= 2\pi abc\int_0^1 (-(-1)+1) r^2\ dr$$
$$= 4\pi abc\int_0^1 r^2\ dr$$
$$= 4\pi abc\frac{1}{3}(1-0) = \frac{4}{3}\boldsymbol{\pi abc}$$

Prema tome

$$\boldsymbol{\Phi} = \frac{4}{3}\boldsymbol{\pi abc}\,.$$

(Zadaci su skinuti sa stranice: \pf.unze.ba\nabokov
Za uočene greške pisati na **infoarrt@gmail.com**)

150 ispitnih zadataka za vježbu podjeljenih po oblastima - detaljno raspisana rješenja ovih zadataka možete skinuti sa stranice pf.unze.ba\nabokov\za_vjezbu

Sadržaj

1 Određeni integrali. Smjena promjenjivih u određenom integralu.

1. Izračunati integrale.

$$(a)\int_2^3 3x^2\mathrm{d}x; \qquad (b)\int_0^4 (1+e^{\frac{x}{4}})\mathrm{d}x; \qquad (c)\int_{-1}^7 \frac{\mathrm{d}t}{\sqrt{3t+4}}; \qquad (d)\int_0^{\frac{\pi}{2a}} (x+3)\sin ax\mathrm{d}x.$$

2. Izračunati integrale.

$$(a)\int_0^5 \frac{x\mathrm{d}x}{\sqrt{1+3x}}; \qquad (b)\int_{\ln 2}^{\ln 3} \frac{\mathrm{d}x}{e^x-e^{-x}}; \qquad (c)\int_1^{\sqrt{3}} \frac{(x^3+1)\mathrm{d}x}{x^2\sqrt{4-x^2}}; \qquad (d)\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{2+\cos x}.$$

3. Dokazati da za parnu funkciju $f(x)$ vrijedi

$$\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_0^a f(x)\mathrm{d}x$$

dok za neparnu funkciju $f(x)$ vrijedi $\int_{-a}^{a} f(x)\mathrm{d}x = 0.$

2 Primjena određenog integrala

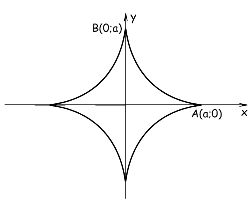
4. Izračunati dužinu luka polukubičnog paraboloida $y^2 = (x-1)^3$ između tački $A(2;-1)$ i $B(5;-8)$.

5. Izračunati dužinu luka jednog svoda cikloide $x = a(t-\sin t)$, $y = a(1-\cos t)$ (za jedan svod cikloide parametar t uzima vrijednosti od 0 do 2π).

6. Izračunati zapreminu tijela koje nastaje rotacijom krive $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ oko y-ose.

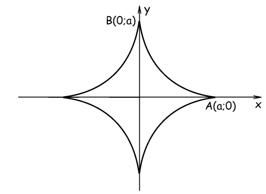
7. Figura u ravni ograničena parabolom $y = 4-x^2$ i poluravnima $y \geq 3$, $y \geq 0$ rotira oko x-ose. Izračunati zapreminu dobijenog tijela.

8. Izračunati površinu omotača tijela koje nastaje kada dio krive $y = x^3$, koji se nalazi između pravih $x = -\frac{2}{3}$ i $x = \frac{2}{3}$, rotira oko x-ose.



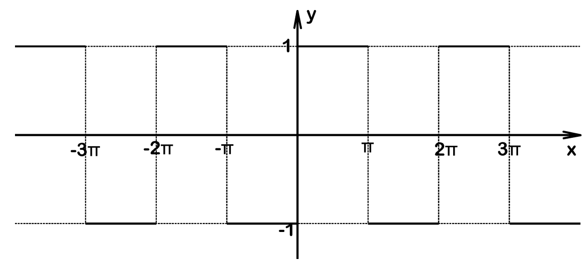
9. Izračunati površinu omotača tijela koje nastaje kada astroida $x = a\cos^3 t$, $y = a\sin^3 t$ rotira oko x-ose (grafik astroide je prikazan na slici lijevo).

10. Figura u ravni ograničena linijama $2y = x^2$ i $2x+2y-3=0$ rotira oko x-ose. Izračunati zapreminu dobijenog tijela.



11. Izračunati zapreminu tijela koje nastaje rotacijom krive $x = a\cos^3 t$, $y = a\sin^3 t$ oko x-ose (data kriva je poznata pod imenom astroida i njen grafik je prikazan na slici lijevo).

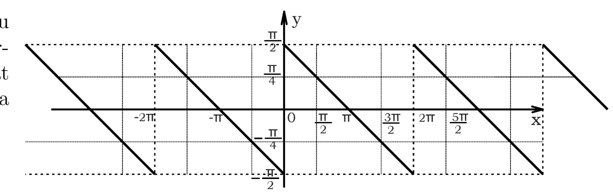
3 Furijeovi redovi



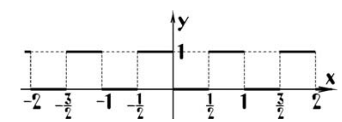
12. Funkciju definisanu grafikom pretvoriti u Furijer-ov red.

Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$.

13. Funkciju definisanu grafikom pretvoriti u Furijer-ov red. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{k=1}^{\infty}\frac{1}{(4n-1)(4n-3)}$.

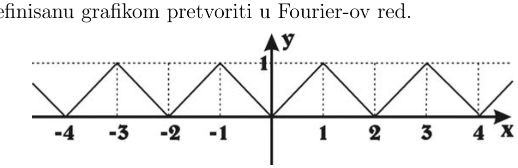


14. Pretvoriti u Fourier-ov red funkciju definisanu grafikom:

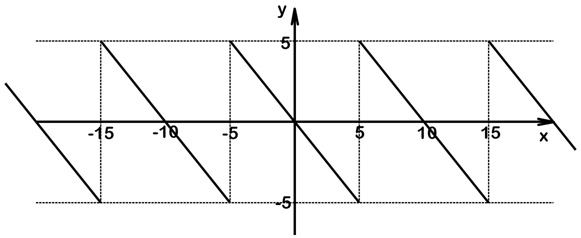


Iskoristiti dobijeni rezultat za izračunavanje sume redova $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$ i $\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}$.

15. Funkciju definisanu grafikom pretvoriti u Fourier-ov red.



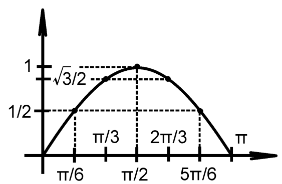
Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty}\frac{1}{2n-1}$.



16. Funkciju definisanu grafikom pretvoriti u Furijer-ov red. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{k=1}^{\infty}\frac{(-1)^n}{n}sin\frac{n\pi}{50}$.

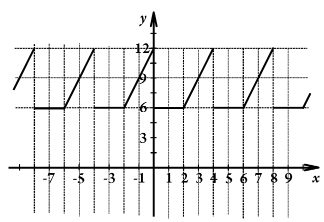
17. Razviti funkciju $f(x)=x(\frac{\pi}{2}-x)$ po sinusima višestrukih uglova u intervalu $(0,\frac{\pi}{2})$.

18. Razviti funkciju $f(x)=\frac{3x^2-6\pi x+2\pi^2}{12}$ u red po kosinusima u intervalu $(0,\pi)$.

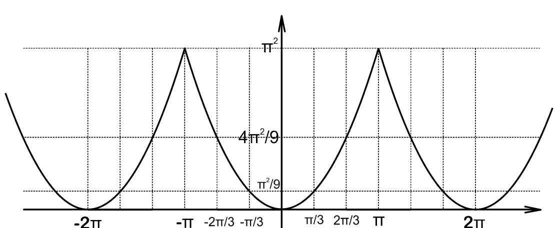


19. Dio grafika f-je $y=f(x)$ je prikazan na slici lijevo. Datu funkciju pretvoriti u Furijer-ov red samo po cos-inusima. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty}\frac{(-1)^n}{1-4n^2}$.

20. Funkciju definisanu grafikom pretvoriti u Furijer-ov red. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$.



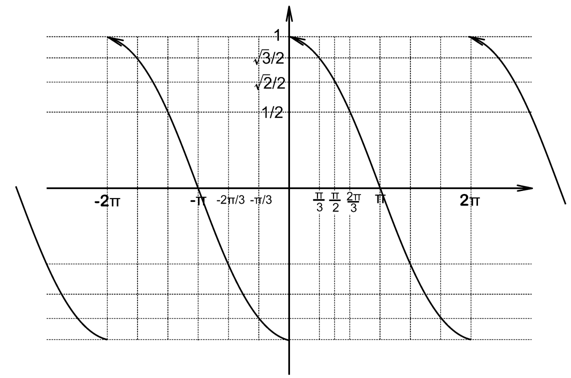
21. Razviti funkciju $f(x)=\frac{3x^2-6\pi x+2\pi^2}{12}$ u red po kosinusima u intervalu $(0,\pi)$. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty}\frac{1}{n^2}$.



22. Funkciju definisanu grafikom razviti u Furijer-ov red. Dobijeni rezultat iskoristiti za sumiranje redova

(a) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+...;$

(b) $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....$



23. Funkciju definisanu grafikom razviti u Furijer-ov red. Dobijeni rezultat iskoristiti za sumiranje reda

$$\frac{1}{1\cdot3}-\frac{3}{5\cdot7}+...+\frac{n\sin\frac{n\pi}{2}}{(2n-1)(2n+1)}+....$$

4 Granične vrijednosti funkcija dviju promjenjivih

24. Neka je data funkcija $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ definisana na sljedeći način

$$f(x,y)=\begin{cases}\frac{(xy)^2}{(xy)^2+(x-y)^2}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$$

Odrediti da li sljedeći limesi postoje i izračunati one limese koji postoje:

(a) $\lim_{x\to0}[\lim_{y\to0}f(x,y)]$; $\lim_{y\to0}[\lim_{x\to0}f(x,y)]$;

(b) $\lim_{(x,y)\to(0,0)}f(x,y)$.

5 Neprekidnost funkcija dvije promjenjive

25. Ispitati neprekidnost funkcije $f(x,y)=\begin{cases}\frac{x^2y^3}{2x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$.

26. Ispitati neprekidnost funkcije $f(x,y)=\begin{cases}\frac{(x-1)^2lnx}{(x-1)^2+y^2}, & (x,y)\neq(1,0)\\ 0, & (x,y)=(1,0)\end{cases}$.

6 Diferencijalni račun funkcija više realnih promjenjivih

27. Ako je $u=\frac{\varphi(x-y)+\psi(x+y)}{x}$ gdje su φ i ψ diferencijalne funkcije, izračunati

$$\frac{\partial}{\partial x}(x^2\frac{\partial u}{\partial x})-x^2\frac{\partial^2u}{\partial y^2}.$$

28. Ako je $z=\frac{y}{f(x^2-y^2)}$ gdje je f diferencijalna funkcija, izračunati $\frac{1}{x}\cdot\frac{\partial z}{\partial x}+\frac{1}{y}\cdot\frac{\partial z}{\partial y}$.

29. Ako je $z = e^y \varphi(ye^{\frac{x^2}{2y^2}})$ gdje je φ diferencijabilna funkcija, dokazati da je

$$(x^2 - y^2)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = xyz.$$

30. Provjeriti da li funkcija $z = arc\,tg\frac{x}{y}$, u kojoj je $x = u+v$, $y = u-v$, zadovoljava jednakost

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{u-v}{v^2+u^2}.$$

31. Ako je $f(x) = \arcsin\frac{x}{y}$ gdje je $y = \sqrt{x^2+1}$ provjeriti da li je $\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{1}{x^2+1}$.

32. Ako je $z = \ln(e^x + e^t)$ gdje je $x = t^3$ izračunati $\frac{\partial z}{\partial t}$ i $\frac{\mathrm{d}z}{\mathrm{d}t}$.

33. Provjeriti da li funkcija $u = \sin x + F(\sin y - \sin x)$, u kojoj je F diferencijabilna funkcija, zadovoljava jednakost $\frac{\partial u}{\partial y}\cos x + \frac{\partial u}{\partial x}\cos y = \cos x \cos y$.

34. Provjeriti da li funkcija $z = \varphi(x^2+y^2)$, u kojoj je φ diferencijabilna funkcija, zadovoljava jednakost

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0.$$

35. Ako je $p = u^2 \ln v$ pri čemu je $u = \frac{x}{y}$ i $v = 3x - 2y$, odrediti $\frac{\partial p}{\partial x}$ i provjeriti da li vrijedi $\frac{\partial p}{\partial y} = -\frac{2xu}{vy^2}(v \ln v + y)$.

7 Tejlorova formula za funkcije dvije i veše promjenjivih

36. Razložiti funkciju $f(x,y) = arc\,tg(x^2y - 2e^{x-1})$ po formuli Tejlora u okolini tačke $M(1,3)$ do stepena drugog reda zaključno.

37. Funkciju $f(x,y) = arc\,tg\frac{x-y}{1+xy}$ razviti u Tejlorov red do članova četvrtog reda u okolini tačke $(0,0)$. Prikazati izgled opšteg člana.

8 Jednačina tangentne ravni i jednačina normale na površ

38. Odrediti jednačinu tangentne ravni na površ $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, koja je normalna na pravu $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$.

39. Naći jednačinu tangentne ravni elipsoida $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ koja na koordinatnim osama odsjeca jednake pozitivne odsječke.

40. Dokazati da tangentne ravni povrsi $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ $(a > 0)$ odsjecaju od koordinatnih osa odsjecke ciji je zbir jednak a.

41. Napisati jednačinu tangentne ravni i normale na površ $2^{\frac{x}{z}} + 2^{\frac{y}{z}} = 8$ u tački $M(2,2,1)$.

9 Izvod funkcije u datom smijeru i gradijent funkcije

42. Izračunati izvod funkcije $u = x^2y^2 + z^2 - 3xyz$ u tački $T(1,1,2)$ u smjeru koji čini s koordinatnim osama uglove $\frac{\pi}{3}$, $\frac{\pi}{4}$ i $\frac{\pi}{6}$.

10 Ekstremi funkcija dvije i više promjenjih

43. Odrediti ekstreme funkcije $f(x,y) = x^2 - xy + y^2 - 2x - 2y$.

44. Naći ekstreme funkcije $z = x + y + 4 + 4\,sin\,x\,sin\,y$.

45. Naći ekstreme funkcije $z = (2x^2 + 3y^2)e^{-(x^2+y^2)}$.

46. Odrediti ekstreme funkcije $f(x,y) = xe^{y+x\sin y}$.

47. Naći ekstreme funkcije $z = x^3 + 4x^2y + xy^2 - 12xy - 3y^2$.

11 Dvostruki integrali

48. Izmjeniti poredak integracije u integralu $\int_0^1 \mathrm{d}y \int_y^{3y} f(x,y)\,\mathrm{d}x$.

49. Izmjeniti poredak integracije u integralu $\int_0^1 \mathrm{d}x \int_{x^3}^{x^2} f(x,y)\mathrm{d}y$.

50. Dati dvostruki integral $\int_0^R \mathrm{d}x \int_0^{\sqrt{R^2-x^2}} f(x,y)\,\mathrm{d}y$ iz pravougaonih koordinata transformisati na polarne koordinate.

51. Izračunati dvostruki integral $I = \iint_D xy\,\mathrm{d}x\,\mathrm{d}y$, gdje je D oblast ograničena linijama $xy = 1$, $x + y = \frac{5}{2}$.

52. Izračunati $\iint_D \mathrm{d}x\,\mathrm{d}y$, ako je $D:\ y^2 - x^2 = 1$, $x^2 + y^2 = 4$.

53. Izračunati $I = \iint_D (x^2+y^2)\mathrm{d}x\mathrm{d}y$ gdje je D paralelogram sa stranicama $y = x$, $y = x + a$, $y = a$, $y = 3a$ $(a > 0)$.

12 Smjena promjenjivih u dvostrukom integralu

54. Izračunati dvostruki integral dat u polarnim koordinatama $I = \iint_D \rho sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\varphi$ gdje je oblast D

a) kružni sektor, ograničen linijama $\rho = a$, $\varphi = \frac{\pi}{2}$ i $\varphi = \pi$;

b) polukrug $\rho \leq 2acos\varphi$, $0 \leq \varphi \leq \frac{\pi}{2}$;

c) oblast između linija $\rho = 2 + cos\varphi$ i $\rho = 1$ (obavezno nacrtati izgled oblasti D u sve tri slučaja).

55. Izračunati integral $I = \iint\limits_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}} \mathrm{d}x\,\mathrm{d}y$ ako je D oblast data sa $x^2+y^2 \leq 1,\ y \geq 0$.

56. Izračunati dvostruki integral $I = \iint\limits_D \mathrm{d}x\,\mathrm{d}y$ ako je D oblast ogranicena lemniskatom $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

57. Izračunati dvostruki integral $\iint_D (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y$ gdje je $D = \{(x,y) \in \mathbb{R} \mid x^2+y^2 \leq \frac{2}{3}(x+2y)\}$.

58. Dati dvostruki integral $\int\limits_{R/2}^{2R} dy \int\limits_0^{\sqrt{2Ry-y^2}} f(x,y)\,dx$ iz pravougaonih koordinata transformisati na polarne koordinate.

59. Izračunati dvostruki integral $\int\limits_0^{\frac{\sqrt{3}}{2}} dx \int\limits_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\,dy$.

60. Izračunati dvostruki integral $\int\limits_0^{\sqrt{\frac{\pi}{2}}} \mathrm{d}x \int\limits_{-\sqrt{\frac{\pi}{2}-x^2}}^{\sqrt{\frac{\pi}{2}-x^2}} \cos(x^2+y^2)\,\mathrm{d}y$.

61. Izračunati:

(a) dvostruki integral $\int_0^{2\pi} \mathrm{d}\varphi \int_0^a \rho^2 \sin^2\varphi\,\mathrm{d}\rho$;

(b) dvojni integral $\iint\limits_G \frac{xy\sqrt{1-x^2-y^2}}{2x^2+y^2} \mathrm{d}x\mathrm{d}y$ gdje je $G = \{(x,y) : x^2+y^2 \leq 1, x \geq 0, y \geq 0\}$.

62. Izračunati $I = \iint\limits_G \left(x + \frac{y^2}{x^2}\right) \mathrm{d}x\mathrm{d}y$ gdje je $G = \{(x,y) : x^2+y^2-2ax \leq 0, a > 0\}$.

63. Izračunati $\iint\limits_D y\mathrm{d}x\mathrm{d}y$ gdje je $D = \{(x,y) : 1 \leq x^2+y^2 \leq 2x,\ y \geq 0\}$.

64. Izračunati $\iint\limits_D x\mathrm{d}x\mathrm{d}y$ gdje je $D = \{(x,y) : 1 \leq x^2+y^2 \leq 2y,\ x \leq y,\ x \geq 0\}$.

65. Izračunati dvojni integral $I = \iint\limits_D \operatorname{arc\,tg}\frac{y}{x}\,\mathrm{d}x\mathrm{d}y$ gdje je
$D = \{(x,y) : 1 \leq x^2+y^2 \leq 9, \frac{x}{\sqrt{3}} \leq y \leq x\sqrt{3}\}$.

66. Izračunati $\iint\limits_D y\,\mathrm{d}x\mathrm{d}y$ gdje je $D = \{(x,y) : x^2+y^2 \leq 1,\ x^2+y^2 \leq 2x,\ y \geq 0\}$.

13 Trostruki integrali

67. Izračunati trojini integral $I = \iiint\limits_G \frac{1}{(1+z)^3} \mathrm{d}x\mathrm{d}y\mathrm{d}z$, gdje je oblast G u prvom oktantu ograničena ravnima $x+y=1,\ z=x+y,\ x=0,\ y=0,\ z=0$.

68. Izračunati trostruki integral $I = \iiint\limits_D \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ ako je Ω oblast omeđena koordinatnim ravnima i sa ravni $x+y+z=1$.

69. Izračunati trostruki integral $I = \iiint\limits_\Omega z\,dx\,dy\,dz$ ako je Ω oblast ograničena površinama $y=x,\ y=2x,\ 2x=1,\ x^2+y^2+z^2=1,\ z \geq 0$.

14 Računanje trostrukih integrala uvođenjem cilindričnih i sfernih koordinata

70. Uvođenjem cilindričnih koordinata izračunati trostruki integral $J = \iiint\limits_W (x^2+y^2+z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$ gdje je oblast W ograničena površinom $3(x^2+y^2)+z^2=3a^2$.

71. Izračunati trostruki integral $K = \iiint\limits_T y\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$ gdje je oblast T ograničena površinama $y = \sqrt{x^2+z^2}$ i $y=h,\ h>0$.

72. Dati trojni integral $\iiint\limits_\Omega f(x,y,z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$ transformisati na trostruki u cilindričnim koordinatama (sa određenim posebnim granicama integracije) ako je Ω oblast u prvom oktantu ograničen cilindrom $x^2+y^2=R^2$ i ravnima $z=0,\ z=1,\ y=x$ i $y=x\sqrt{3}$.

73. Dat je trostruki integral $\int\limits_0^{2\pi} \mathrm{d}\varphi \int\limits_0^2 r^3\,\mathrm{d}r \int\limits_0^{\sqrt{4-r^2}} \mathrm{d}z$ u cilindričnim koordinatama. Skicirati oblast integracije i izračunati taj integral prelazeći na sferne koordinate.

74. Izračunati trostruki integral $K = \iiint\limits_T y\mathrm{d}x\mathrm{d}y\mathrm{d}z$ gdje je oblast T ograničena površinama $y = \sqrt{x^2+z^2}$ i $y=h,\ h>0$.

75. Izračunati integral

$$\iiint\limits_\Omega \sqrt{x^2+y^2+z^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

gdje je $\Omega = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 \leq z,\ x^2+y^2 \leq z^2\}$.

76. Uvođenjem sfernih koordinata izračunati integral $\int\limits_0^1 \mathrm{d}x \int\limits_0^{\sqrt{1-x^2}} \mathrm{d}y \int\limits_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2\mathrm{d}z$.

77. Izračunati integral $\iiint\limits_V xyz\mathrm{d}x\mathrm{d}y\mathrm{d}z$ gdje je oblast V ograničena sferom $x^2+y^2+z^2=1$ i ravnima $x=0$, $y=0$, $z=0$ u I oktantu.

15 Primjena dvostrukog i trostrukog integrala

78. Izračunati zapreminu tijela, koje je ograničeno sa površinama $z=y^2-x^2$, $z=0$, $y=\pm 2$.

79. Izračunati zapreminu tijela, ograničeno površinama $y=x^2$, $y=1$, $x+y+z=4$, $z=0$.

80. Izračunati zapreminu tijela ograničenog dijelom površi $(x^2+y^2+z^2)^3=\dfrac{a^6z^2}{x^2+y^2}$, $a>0$ u I oktantu.

81. Izračunati zapreminu tijela koje je ograničeno površima $x^2+y^2+z^2=4$ i $x^2+y^2=3z$.

82. Izračunati zapreminu tijela ogranicenog valjkom $x^2+y^2=6x$ i ravnima $x-z=0$, $5x-z=0$.

83. Izračunati zapreminu tijela ograničenog ravninom $x0y$, valjkom $x^2+y^2=2ax$ i čunjem $x^2+y^2=z^2$.

84. Izračunati zapreminu tijela koju ravan $z=x+y$ odsijeca od paraboloida $z=x^2+y^2$.

85. Izračunati zapreminu dijela kugle $x^2+y^2+z^2=R^2$ koji se nalazi između dvije paralelne ravni $z=0$ i $z=a$ $(0<a<R)$.

86. Naći težište homogenog tijela ograničenog sa ravnima $x=0$, $y=0$, $z=0$, $x=2$, $y=4$ i $x+y+z=8$ (koso zasiječen paralelopiped).

87. Izračunati zapreminu tijela ograničenog loptom $x^2+y^2+z^2=a^2$, cilindrom $x^2+y^2=ax$ i ravni $0xy$ koji se nalazi u gornjem poluprostoru.

16 Krivoliniski integral prve vrste (po luku)

88. Izračunati krivoliniski integral $I=\int\limits_L(4\sqrt[3]{x}-3\sqrt{y})dl$ između tački $E(-1;0)$ i $F(0;1)$

a) po pravoj EF;
b) po liniji astroide $x=cos^3t$, $y=sin^3t$.

89. Izračunati krivoliniski integral prve vrste

$$I=\oint\limits_C\sqrt{x^2+y^2}\,\mathrm{d}s$$

gdje je C krug $x^2+y^2=ax$, (a>0).

90. Izračunati krivoliniski integral $\int\limits_L(x-y)\,\mathrm{d}s$ po kružnoj liniji $x^2+y^2=ax$.

91. Izračunati krivoliniski integral prve vrste $\oint\limits_c(x+y)\,dS$ ako je $c:\begin{cases}x=a\,cos\varphi\,\sqrt{cos2\varphi}\\ y=a\,sin\varphi\,\sqrt{cos2\varphi}\\ -\frac{\pi}{4}\le\varphi\le\frac{\pi}{4}\end{cases}$ (kriva c je desna latica lemniskate $\rho=a\sqrt{cos2\varphi}$).

92. Izračunati krivoliniski integral $I=\int\limits_{AB}\dfrac{dl}{\sqrt{x^2+y^2}}$ po odsječku prave $x-2y=4$ od tačke $A(0;-2)$ do tačke $B(4;0)$.

93. Neka je A tačka u kojoj prava $2x-\sqrt{5}y-1=0$ siječe y-osu, a B tačka u kojoj data prava siječe x-osu. Izračunati krivolinijski integral prve vrste $\int\limits_C\dfrac{\mathrm{d}s}{\sqrt{x^2+y^2+1}}$, ako je C odsiječak date prave između tačaka A i B.

17 Krivoliniski integral druge vrste (po koordinatama)

94. Izračunati krivoliniske integrale

a) $\oint\limits_{-l}2x\,dx-(x+2y)\,dy$ i b) $\oint\limits_{+l}ycosx\,dx+sinx\,dy$

po krivoj l, gdje je l trougao čiji su vrhovi $A(-1;0)$, $B(0;2)$ i $C(2;0)$.

95. Date su tačke $A(3;-6;0)$ i $B(-2;4;5)$. Izračunati krivoliniski integral $I=\int\limits_c xy^2\,dx+yz^2\,dy-zx^2\,dz$ gdje je c:
(a) duž koja spaja tačke O i B (O je koordinatni početak)
(b) kriva od A do B kruga zadan jednačinama $x^2+y^2+z^2=45$, $2x+y=0$.

96. Izračunati krivoliniski integral $I=\int\limits_c(x^2+y^2)\,dx+x^2y\,dy$ gdje je c kontura trapeza koga obrazuju prave $x=0$, $y=0$, $x+y=1$ i $x+y=2$.

97. Izračunati krivoliniski integral

$$I=\oint\limits_C z\,dz$$

duž krive koja nastaje kao presjek cilindra $\dfrac{(x-\frac{a}{2})^2}{\frac{a^2}{2}}+\dfrac{(y-\frac{b}{2})^2}{\frac{b^2}{2}}=1$ i paraboloida $z=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$ orjentisana u pozitivnom smijeru $(a\ge b>0)$.

98. Izračunati krivoliniski integral $I=\oint\limits_c y\,dx+x^2\,dy$ duž krive koja nastaje kao presjek ravni $z=0$ i cilindra $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{x}{a}+\dfrac{y}{b}$ orjentisana u pozitivnom smijeru $(a\ge b>0)$.

99. Izračunati integral $I=\oint\limits_c y^2\,dx$ po krivoj koja nastaje kao presjek kugle $x^2+y^2+z^2=R^2$ i valjka $x^2+y^2=Rx$. (Mala pomoć: Da bi ste izračunali ovaj integral treba parametrizirati krivu c. Jedan od načina kako to možete postići je da krenete od parametrizacije kruga...)

100. Izračunati krivoliniske integrale (a) $I=\int\limits_{-l}2xdx-(x+2y)dy$; (b) $I=\int\limits_{+l}ycosxdx+sinxdy$; gdje je l kontura trougla čiji su vrhovi $A(-1;0)$, $B(0;2)$ i $C(2;0)$.

101. Izračunati krivolinijski integral druge vrste

$$I=\oint\limits_C x\mathrm{d}y+x\mathrm{d}z$$

gdje je C kriva koja nastaje presjekom cilindrične površi $x^2+y^2=2x$ i ravni $z=x$ pozitivno orjentisana ako se posmatra iz tačke $(0;0;1)$.

102. Izračunati krivoliniski integral druge vrste $I=\oint\limits_C (y-z)\mathrm{d}x+(z-x)\mathrm{d}y+(x-y)\mathrm{d}z$ gdje je C krug $x^2+y^2+z^2=a^2$ $(a>0)$, $y=x\operatorname{tg}\alpha$, $(0<\alpha<\frac{\pi}{2})$ uzet u smjeru suprotnom kretanju kazaljnke na satu ako se posmatra sa pozitivnog dijela x-ose.

103. Izračunati vrijednost krivoliniskog integrala $I=\oint\limits_C y\mathrm{d}x+z\mathrm{d}y+x\mathrm{d}z$ duž zatovorene krive C koja je dobijena kao presjek sljedećih površina: $x^2+y^2=r^2$ i $x^2=rz$ $(r>0)$. (Kriva C je orjentisana pozitivno ako se posmatra sa z-ose za $z>r$).

18 Green-Gausova formula

104. Pomoću Greenove formule izračunati krivoliniski integral

$$\oint\limits_c (x^2y+\frac{1}{3}y^3+ye^{xy})\,dx+(x+xe^{xy})\,dy$$

ako je c pozitivno rjentisana kontura određena linijama $y=\sqrt{1-x^2}$, $y=0$.

105. Izračunati krivoliniski integral $I=\int\limits_c (xy+x+y)dx+(xy+x-y)dy$ ako je $c:\ x^2+y^2=3x$.

106. Pomoću Greenove formule izračunati integral $I=\int_c (xy+x+y)\,dx+(xy+x-y)\,dy$, ako je c kontura kruga $x^2+y^2=ax$ prijeđena u pozitivnom smislu.

107. Izračunati

$$I=\int\limits_C (e^{x+y}\sin 2y+x+y)\mathrm{d}x+(e^{x+y}(2\cos 2y+\sin 2y)+2x)\mathrm{d}y$$

gdje je C kriva $y=\sqrt{2x-x^2}$, integracija se vrši od tačke $A(2;0)$ do tačke $O(0;0)$.

108. Izračunati

$$I=\int\limits_{\overset{\frown}{AO}} (e^x\sin y-my)\mathrm{d}x+(e^x\cos y-m)\mathrm{d}y$$

gdje je $\overset{\frown}{AO}$ gornji polukrug $x^2+y^2=ax$, $y\geq 0$ $(a>0)$ orjentisan od tačke $A(a;0)$ do tačke $O(0;0)$.

19 Primjena krivoliniskog integrala druge vrste: Računanje površine ravne figure

109. Uz pomoć krivoliniskog integrala druge vrste, izračunati površinu, ograničenu kardioidom $x=2\,cost-cos2t$, $y=2sint-sin2t$.

110. Izračunati pomoću krivoliniskog integrala druge vrste površinu ravne figure ograničene konturom

$$\text{c}:\begin{cases} x=a(t-sin\,t)\\ y=a(1-cos\,t)\\ 0\leq t\leq 2\pi\end{cases}$$

20 Nezavisnost krivoliniskog integrala od vrste konture. Određivanje primitivnih funkcija

111. Izračunati krivoliniski integral $\int\limits_{(1,0)}^{(6,8)} \frac{x\,dx+y\,dy}{\sqrt{x^2+y^2}}$ duž puta koji ne prolazi kroz koordinatni početak.

112. Izračunati krivoliniski integral $\int\limits_{(2,1)}^{(1,2)} \frac{y\,dx-x\,dy}{x^2}$ duž puta koji ne siječe osu $0y$.

21 Površinski integral prve vrste

113. Izračunati površinski integral $I=\iint\limits_S xyz\,dS$, ako je S dio ravni $x+y+z=1$ u I oktantu.

114. Izračunati površinski integral $\iint_S 3z\,dS$ gdje je S površina paraboloida $z=2-(x^2+y^2)$ iznad xy-ravni.

115. Izračunati površinski integral

$$\iint\limits_{(S)} \sqrt{-x^2+4}\,dS,$$

gdje je (S) omotač površi $\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}$, $0\leq z\leq 3$.

116. Izračunati površinski integral prvog tipa $I=\iint\limits_W (x^2+y^2)\mathrm{d}s$ gdje je W-površina dijela paraboloida $x^2+y^2=2z$ koju odsjeca ravan $z=1$ (dio paraboloida ispod date ravni).

117. Izračunati površinski integral $I=\iint\limits_S \frac{\mathrm{d}S}{(1+z)^2}$ ako je S sfera $x^2+y^2+z^2=1$.

22 Površinski integral druge vrste

118. Izračunati površinski integral drugog tipa (po koordinatama) $I=\iint\limits_\sigma \sqrt[4]{x^2+y^2}dxdy$ gdje je σ donja strana kruga $x^2+y^2\leq a^2$.

119. Izračunati površinski integral

$$K=\iint\limits_{-W} y\,dx\,dz$$

gdje je W-površina tetraedra ograničenog ravnima $x+y+z=1$, $x=0$, $y=0$ i $z=0$.

120. Izračunati površinski integral $\iint\limits_T 2\,dxdy + y\,dxdz - x^2z\,dydz$ gdje je T vanjska strana elipsoida $4x^2 + y^2 + 4z^2 = 4$ koji se nalazi u prvom oktantu.

121. Izračunati površinski integral $\iint\limits_S xy^3z\,dx\,dy$ ako je S vanjska strana sfere $x^2+y^2+z^2 = 4$ u prvom oktantu.

122. Izračunati površinski integral druge vrste

$$I = \iint\limits_S xyz\,\mathrm{d}x\mathrm{d}y$$

gdje je S spoljna strana dijela sfere $x^2 + y^2 + z^2 = 1$, $x \geq 0$, $y \geq 0$.

123. Izračunati površinski integral drugog tipa (po koordinatama) $I = \iint\limits_\sigma \sqrt[4]{x^2+y^2}\mathrm{d}x\mathrm{d}y$ gdje je σ donja strana kruga $x^2 + y^2 \leq a^2$.

124. Izračunati

$$I = \iint\limits_{S+} \left(\frac{1}{x}\mathrm{d}y\mathrm{d}z + \frac{1}{y}\mathrm{d}z\mathrm{d}x + \frac{1}{z}\mathrm{d}x\mathrm{d}y\right)$$

gdje je $S+$ spoljašnja strana jedinične sfere (zadatak uraditi bez upotrebe teoreme Gauss-Ostrogradskog - zadatak se i ne može uraditi uz pomoć navedene teoreme zato što ne ispunjavaju sve uslove teoreme).

125. Izračunati površinski integral $I = \iint\limits_{S+} y^2\mathrm{d}y\mathrm{d}z + (y^2 + x^2)\mathrm{d}z\mathrm{d}x + (y^2 + x^2 + z^2)\mathrm{d}x\mathrm{d}y$ gdje je $S+$ spoljašnja strana polusfere $x^2 + y^2 + z^2 = 2Rx$, $z > 0$ (za fiksirano $R > 0$).

126. Data je kriva c koja je dobijena kao presjek površina $x^2 + y^2 = r^2$ i $x^2 = rz$ $(r > 0)$. Izračunati površinski integral $\iint\limits_S \mathrm{d}x\mathrm{d}y$ gdje je S gornja strana površine koju zatvara kriva c.

23 Primjena površinskog integrala

127. Izračunati $\iint_S dS$, ako je S površina djela sfere $S = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = a^2\}$ koja se nalazi u unutrašnjosti cilindra $S_1 = \{(x,y,z) \in \mathbb{R}^3 \,|\, \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, z \in \mathbb{R}\}$, $b < a$.

128. Neka je S površina tijela koje je dobijeno presjekom dva cilindra $S_1 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + z^2 = a^2, y \in \mathbb{R}\}$ i $S_1 = \{(x,y,z) \in \mathbb{R}^3 \,|\, y^2 + z^2 = a^2, x \in \mathbb{R}\}$.
Izračunati $\iint\limits_S dS$.

129. Izračunati površinu dijela površi $S : z^2 = 2xy$ određene u prvom oktantu u presjeku sa ravnima: $x = 0$, $y = 0$ i $x + y = 1$.
Uputa: $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$, $B(\frac{3}{2},\frac{3}{2}) = \frac{\pi}{8}$, $B(\frac{1}{2},\frac{5}{2}) = \frac{3\pi}{8}$.

130. Izračunati površinu dijela lopte $x^2+y^2+z^2 = 3a^2$ koja se nalazi ispod parabole $x^2+y^2 = 2az$ a iznad $x0y$ ravni.

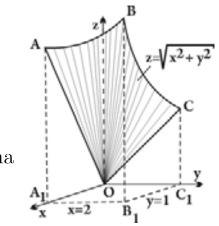
131. Izračunati površinu onog dijela kupe $z^2 = x^2+y^2$ koji se nalazi unutar valjka $x^2+y^2 = 2x$.

132. Odrediti površinu koju cilindar $x^2 + y^2 = ax$ isjeca na lopti $x^2 + y^2 + z^2 = a^2$ iznad ravni Oxy.

24 Formula Stoksa

133. Uz pomoć formule Stoksa, izračunati krivoliniski integral

$$\oint\limits_l e^x dx + z(x^2+y^2)^{\frac{3}{2}}dy + yz^3dz$$



gdje je l-zakrivljena linija OCBAO (vidi sliku) dobijena presjekom površina $z = \sqrt{x^2+y^2}$, $x = 0$, $x = 2$, $y = 0$, $y = 1$.

134. Uz pomoć formule Stoksa, izračunati krivolinijski integral $I = \oint\limits_L x^2y^3\mathrm{d}x + \mathrm{d}y + z\mathrm{d}z$ gdje je L krug dat sa $x^2 + y^2 = r^2$ i $z = 0$ (r>0). (L je pozitivno orjentisana kriva ukoliko se posmatra sa pozitivnog dijela z-ose.)

25 Formula Gaus-Ostrogradskog

135. Uz pomoć formule Gauss-Ostrogradski izračunati površinski integral

$$\oiint\limits_S 4x^3\mathrm{d}y\mathrm{d}z + 4y^3\mathrm{d}x\mathrm{d}z - 6z^4\mathrm{d}x\mathrm{d}y$$

gdje je S vanjska strana cilindra $x^2 + y^2 = a^2$ koji se nalazi između ravni $z = 0$ i $z = h$.

136. Pomoću formule Gauss-Ostrogradski izračunati površinski integral

$$I = \oiint\limits_S xz\,\mathrm{d}y\mathrm{d}z + xy\,\mathrm{d}z\mathrm{d}x + yz\,\mathrm{d}x\mathrm{d}y,$$

ako je S vanjska strana tijela koje pripada prvom oktantu i ograničeno je cilindrom $x^2+y^2 = 1$, te ravnima $x = 0$, $y = 0$, $z = 0$, $z = 2$.

137. Izračunati površinski integral $I = \iint\limits_{S+} x^2\mathrm{d}y\mathrm{d}z + y^2\mathrm{d}z\mathrm{d}x + z^2\mathrm{d}x\mathrm{d}y$ gdje je $S+$ spoljašnja strana kupe određena omotačem $z^2 = x^2 + y^2$, $0 \leq z \leq h$ i osnovom $x^2 + y^2 \leq h^2$, $z = h$ za fiksirano $h > 0$.

26 Integrali ovisni o parametru

138. Prvo izračunati integral $I = \int\limits_0^{\infty} e^{-x}\sin(\alpha x)\mathrm{d}x$ pa poslije toga dobijeni rezultat iskoristiti i koristeći metodu diferenciranja po parametru izračunati

$$G(\alpha) = \int\limits_0^{\infty} xe^{-x}\cos(\alpha x)\mathrm{d}x$$

139. Date su vrijednosti dva integrala ($\alpha > 0$)

$$\int_0^{\infty} \frac{\cos\alpha x}{1+x^2}\mathrm{d}x = \frac{\pi}{2}e^{-\alpha}, \qquad \int_0^{\infty} \frac{\sin\alpha x}{x}\mathrm{d}x = \frac{\pi}{2}.$$

Koristeći date jednakosti, uz pomoć metode diferenciranja po parametru izračunati $\int_0^{\infty} \frac{\sin\alpha x}{x(1+x^2)}\mathrm{d}x$.

140. Metodom diferenciranja po parametru izračunati integral $\int\limits_0^1 \frac{\ln(1-a^2x^2)}{x^2\sqrt{1-x^2}}\mathrm{d}x \quad (a^2 < 1)$ (mala pomoć: možda ćete naći korisno da u rješavanju integrala iskoristie smjene $x = \sin t$ ili $\operatorname{tg} t = z$).

141. Metodom diferenciranja po parametru izračunati integral $\int\limits_0^1 \frac{\operatorname{arc\,tg} ax}{x\sqrt{1-x^2}}\mathrm{d}x$ (mala pomoć: možda ćete naći korisno da u rješavanju integrala iskoristie smjene $x = \sin t$ ili $\operatorname{tg} t = z$).

27 Vektorska teorija polja

142. Dokazati da je vektorsko polje potencijalno i naći njegov potencijal:

$$\vec{v} = 2x(y^2+z^2)\vec{i} + 2y(x^2+z^2)\vec{j} + 2z(x^2+y^2)\vec{k}.$$

143. Odrediti brojeve a i b tako da vektorsko polje $\vec{v} = (yz + axy, xz + bx^2 + yz^2, axy + y^2z)$ bude potencijalno i za dobijeno polje izračunati njegovu cirkulaciju duž pravoliniske konture od tačke $A(1;1;1)$ prema tački $B(2;2;2)$

144. Neka funkcije $g, h : \mathbb{R}^3 \longrightarrow \mathbb{R}$ ispinjavaju

$$\Delta g(x,y,z) = 0 \quad \text{i} \quad \Delta h(x,y,z) = 0$$

gdje je $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ Laplace-ov operator. Za funkciju $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ datu sa

$$f(x,y,z) = g(x,y,z) + (x^2+y^2+z^2)h(x,y,z)$$

izračunati $\Delta\Delta f(x,y,z)$.

145. Pokazati da je vektorsko polje $\vec{v} = (2x+y+z, x+2y+z, x+y+2z)$ potencijalno i naći njegov potencijal.

146. Dokazati da je vektorsko polje $\vec{v} = (z\cos zx - y\sin x, \cos x, x\cos zx)$ potencijalno i izračunati cirkulaciju tog polja duž prave od tačke $O(0,0,0)$ do tačke $A(1,2,\pi)$.

28 Cirkulacija i fluks vektorskog polja

147. Izračunati cirkulaciju vektorskog polja $\vec{v} = (1, xy^2, yz^2)$ duž konture $x^2 + 2y^2 = 4$, $z = 2x$.

148. Izračunati cirkulaciju polja $\vec{v} = x\vec{i} + y\vec{j} + (x+y-1)\vec{k}$ duž odsječka prave između tačaka $A(1,1,1)$ i $B(2,3,4)$.

149. Data su skalarna polja $f = xyz$, $g = xy + yz + zx$.

(a) Formirati vektorska polja $\vec{a} = \operatorname{grad} f$, $\vec{b} = \operatorname{grad} g$ i ispitati prirodu vektorskog polja $\vec{a} \times \vec{b}$ (drugim riječima odgovoriti na pitanje da li je polje $\vec{a} \times \vec{b}$ potencijalno ili solenoidno).

(b) Izračunati $\int\limits_C (\vec{a} \times \vec{b})\mathrm{d}r$, gdje je C duž koja spaja tačke $O(0,0,0)$ i $B(1,2,3)$.

150. Izračunati fluks vektorskog polja

$$\vec{v} = (x, -y^2, x^2+z^2-1)$$

po unutrašnjoj strani sfere $x^2 + y^2 + z^2 = 1$.